King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260 Final Exam
Semester I, 2012 (121)

Name:	
ID :	
Serial no.:	
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Q		Points
1		9
2		15
3		15
4		15
5		8
6		15
7		8
8		8
9		8
10		8
11		8
12		8
13		15 (3 each)
Total		**140**

(1) Determine a form for a particular solution y_p of the differential equation. **DO NOT determine the coefficients**.

$$y''' - 3y'' + 7y' - 5y = x + e^x$$

[9 points]

Solution: ① Homogeneous equation: $y''' - 3y'' + 7y' - 5y = 0$.

(*) Characteristic Equation: $r^3 - 3r^2 + 7r - 5 = 0$

$\Longrightarrow r^3 - r^2 - 2r^2 + 2r + 5r - 5 = 0$

$\Longrightarrow r^2(r-1) - 2r(r-1) + 5(r-1) = 0$

$\Longrightarrow (r-1)(r^2 - 2r + 5) = 0$

$\Longrightarrow (r-1)(r^2 - 2r + 1 + 4) = 0$. So $(r-1)((r-1)^2 - (2i)^2) = 0$

Then $(r-1)(r-1-2i)(r-1+2i) = 0$. The roots are

$r = 1$, $r = 1 + 2i$ and $\bar{r} = 1 - 2i$

The complementary solution is $\boxed{y_c = c_1 e^x + c_2 e^x \cos 2x + c_3 e^x \sin 2x}$

3

② The particular solution:

Dividing the equation in two subequations:

(E_1): $y''' - 3y'' + 7y' - 5y = x \Longrightarrow y_{p_1} = A_0 + A_1 x$

(E_2): $y''' - 3y'' + 7y' - 5y = e^x \Longrightarrow y_{p_2} = B_0 x e^x$

The particular solution is $\boxed{y_p = A_0 + A_1 x + B_0 x e^x}$

6

(2)

Given $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ [15 points]

(a) Find the eigenvalues of A .

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Longrightarrow (2-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2, \quad \lambda_3 = 1$$

The eigenvalues are $\lambda = 2, 2, 1$ 3

(b) Find the corresponding eigenvectors.

For $\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow k_3 = 0, \; k_1 = -k_2$

take $k_2 = 1 \Rightarrow k_1 = -1$

So the Eigenspace E_1 has a basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

2

For $\lambda = 2$ $\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow k_2 = k_3, \; k_1$ free

If we take $k_1 = 0, \; k_2 = k_3 = 1$ we get $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. If we take $k_1 = 1, \; k_2 = k_3 = 0$ we get $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

So, a basis for E_2 is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

4

(c) Is A diagonalizable?

Yes, Since $\dim(E_1) + \dim(E_2) = 3$

$1 + 2 = 3$ 2

(d) Find a diagonal matrix B and an invertible matrix P such that $B = P^{-1}AP$.

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}, \quad \alpha = \{\alpha_1, \alpha_2, \alpha_3\}$$

2

$$\left.\begin{aligned} A\alpha_1 &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow [A\alpha_1]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ A\alpha_2 &= 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow [A\alpha_2]_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \\ A\alpha_3 &= 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [A\alpha_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{aligned}\right\} \Rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2

(3) Find the general solution of $Y' = AY$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

[15 points]

Solution. $\begin{vmatrix} \lambda-1 & 0 & 0 \\ -2 & \lambda-1 & 2 \\ -3 & -2 & \lambda-1 \end{vmatrix} = (\lambda-1)\left((\lambda-1)^2+4\right)$. Eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1+2i$, $\lambda_3 = 1-2i$. 3

$\lambda_1 = 1$: $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}$, basis for E_1: $\left\{ \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\}$ 2

$\lambda_2 = 1+2i$: $\begin{pmatrix} 2i & 0 & 0 \\ -2 & 2i & 2 \\ -3 & -2 & 2i \end{pmatrix}$, basis for E_{1+2i}: $\left\{ \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} \right\}$ 2

Eigenvalue $\lambda_1 = 1$ gives solution $P\begin{bmatrix} e^x \\ 0 \\ 0 \end{bmatrix}$ where $P = \begin{bmatrix} 2 & 0 & 0 \\ -3 & i & -i \\ 2 & 1 & 1 \end{bmatrix}$, i.e. $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^x$. 2

Eigenvalue $\lambda_2 = 1+2i$ gives solution $P\begin{bmatrix} 0 \\ e^{(1+2i)x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -e^x \sin(2x) \\ e^x \cos(2x) \end{bmatrix} + i \begin{bmatrix} 0 \\ e^x \cos(2x) \\ e^x \sin(2x) \end{bmatrix}$

Then $\begin{bmatrix} 0 \\ -\sin(2x) \\ \cos(2x) \end{bmatrix} e^x$ and $\begin{bmatrix} 0 \\ e^x \cos(2x) \\ e^x \sin(2x) \end{bmatrix} e^x$ are linearly independent solutions 4 of $Y' = AY$. General solution

$$Y = C_1 \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^x + C_2 \begin{bmatrix} 0 \\ -\sin(2x) \\ \cos(2x) \end{bmatrix} e^x + C_3 \begin{bmatrix} 0 \\ \cos(2x) \\ \sin(2x) \end{bmatrix} e^x .$$ 2

(4)

Solve the system: $Y' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{pmatrix} Y$.

[15 points]

Solution:

① Eigenvalues and Eigenvectors.

$$P_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & 1 \\ 0 & \lambda-1 & 0 \\ 1 & 1 & \lambda-2 \end{vmatrix}$$ Expanding along with the second row:

$$P_A(\lambda) = (\lambda-1)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} = (\lambda-1)\left[(\lambda-2)^2-1\right]$$
$$= (\lambda-1)(\lambda-2-1)(\lambda-2+1) = (\lambda-1)(\lambda-3)(\lambda-1)$$
$$= (\lambda-1)^2(\lambda-3).$$ The eigenvalues are $\lambda = 1, \lambda = 3$. 2

(*) Eigenvectors for $\lambda = 1$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = c \\ b = c-a \\ = 0 \end{matrix}$

$$E_{\lambda=1} = \left\{ \begin{pmatrix} c \\ 0 \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3 Let $V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$: "There is one missing vector" by solving:

Find the second "missing" vector V_2

$(\lambda I - A)V_2 = V_1$ that is $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \left.\begin{matrix} c = 1+a \\ c = a+b-1 \end{matrix}\right\} \Rightarrow \begin{matrix} a = c-1 \\ b = 1+c-a = 2 \end{matrix}$$ Taking $c = 0$,

we may assume that $V_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ 3

The corresponding part of Solution are:

$$Y_1 = e^x V_1 = e^x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } Y_2 = e^x\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right) = e^x\begin{pmatrix} 1-x \\ 2x \\ 1 \end{pmatrix}$$

(*) Eigenvalues fo $\lambda = 3$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. So $\begin{matrix} b = 0 \\ a = -c \end{matrix}$

The corresponding part of solution is :

$$E_{\lambda=3} = \left\{ \begin{pmatrix} -c \\ 0 \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Y_3 = e^{3x} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The General Solution is $Y = \begin{bmatrix} e^x & (1-x)e^x & -e^{3x} \\ 0 & 2x\,e^x & 0 \\ e^x & e^x & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

or $Y = c_1 Y_1 + c_2 Y_2 + c_3 Y_3$

Thus $Y = \begin{pmatrix} c_1 e^x + c_2 (1-x) e^x - c_3 e^{3x} \\ 2c_2 x e^x \\ c_1 e^x + c_2 e^x + c_3 e^{3x} \end{pmatrix}$ 4

(5) Convert the following linear differential equation to a system of linear equations

$$y'' - 2y' + 3y = \tan x$$

[8 points]

<u>Soln</u>: Let $v_1 = y$ and $v_2 = y'$ this gives 2

$$\begin{cases} v_1' = y' = v_2 & 1 \\ v_2' = y'' = 2y' - 3y + \tan x = 2v_2 - 3v_1 + \tan x & 1 \end{cases}$$

which can be written as

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tan x \end{bmatrix} .$$ 4

(6) Use the Method of Variation of Parameters to find the general solution of

$$y'' - 2y' + 2y = e^x \tan x$$

[15 points]

Soln: $y'' - 2y' + 2y = 0$

$m^2 - 2m + 2 = 0 \Rightarrow m = 1 \mp i$ 2

$\Rightarrow y_c = c_1 e^x \cos x + c_2 e^x \sin x$

$y_1 = e^x \cos x \qquad y_2 = e^x \sin x$ 2

$$W(y_1, y_2) = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x \end{vmatrix} = e^{2x}$$ 2

$$u_1' = \frac{W_1}{W} = -\frac{(e^x \sin x)(e^x \tan x)}{e^{2x}} = -\frac{\sin^2 x}{\cos x}$$

$$= \cos x - \sec x$$

$u_1 = \sin x - \ln|\sec x + \tan x|$ 2

$$u_2' = \frac{W_2}{W} = \frac{(e^x \cos x)(e^x \tan x)}{e^{2x}} = \sin x \Rightarrow u_2 = -\cos x$$ 2

$$y_p = \left[\sin x - \ln|\sec x + \tan x|\right] e^x \cos x - (\cos x) e^x \sin x$$

$$= -(e^x \cos x) \ln|\sec x + \tan x|$$ 3

2

$$y = c_1 e^x \cos x + c_2 e^x \sin x - (e^x \cos x) \ln|\sec x + \tan x|$$

(7) Solve the Bernoulli equation

$$xy' + 2y = 4x^4y^4 \,, \qquad x > 0$$

[8 points]

Soln:

$$y' + \frac{2}{x}\, y = 4x^3 y^4$$

2

let $w = y^{-3} \Rightarrow \frac{dw}{dx} = -3\, y^{-4} \frac{dy}{dx}$

$$\Rightarrow -\frac{1}{3}\frac{dw}{dx} + \frac{2}{x}\, w = 4x^3$$

or $\frac{dw}{dx} - \frac{6}{x}\, w = -12\, x^3$ 2

I. factor $u(x) = e^{\int -\frac{6}{x}dx} = x^{-6}$ 2

$$\Rightarrow x^{-6}\frac{dw}{dx} - 6\, x^{-7}\, w = -12\, x^{-3}$$

or $\frac{d}{dx}\left(x^{-6} w \right) = -12\, x^{-3}$

$\Rightarrow x^{-6} w = 6\, x^{-2} + C$ or $w = 6\, x^4 + C\, x^6$

$\Rightarrow y^{-3} = 6x^4 + c\, x^6 \Rightarrow y^3 = \dfrac{1}{x^4(6 + c\, x^2)}$

2

(8) Given the DE [8 points]

$$\left[\sin(x-y)+\cos(x+y)+y\right]dx-\left[\sin(x-y)-\cos(x+y)-x\right]dy=0$$

a) Show that it is Exact.

$$\frac{\partial M}{\partial y} = -\cos(x-y) - \sin(x+y) + 1 = \frac{\partial N}{\partial x}$$

2

b) Solve it.

$$\frac{\partial f}{\partial x} = M(x,y) = \sin(x-y) + \cos(x+y) + y \quad -(1)$$

and

$$\frac{\partial f}{\partial y} = N(x,y) = -\sin(x-y) + \cos(x+y) + x \quad -(2)$$

From (1) $\quad f(x,y) = -\cos(x-y) + \sin(x+y) + xy + g(y)$

2

using (2),

$$\frac{\partial f}{\partial y} = -\sin(x-y) + \cos(x+y) + x + g'(y)$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C_1$$

So, the solution is

4

$$\sin(x+y) - \cos(x-y) + xy = C$$

(9) If y is a solution of the IVP $y'' - 5y' + 4y = 0$; $y(0) = 0$, $y'(0) = 3$, then $y(\ln 2) =$

[8 points]

(a) 10
(b) 12
(c) 14
(d) 16
(e) 18

(10) Which of the following is a fundamental set of solutions of the DE $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$?

[8 points]

(a) $\left\{ \frac{1}{x} , x , x^3 \right\}$

(b) $\left\{ \frac{1}{x} , x , x^2 \right\}$

(c) $\left\{ \frac{1}{x} , x , 2x - \frac{3}{x} \right\}$

(d) $\left\{ \frac{1}{x} , x , e^x \right\}$

(e) $\left\{ \frac{1}{x} , x , 1 \right\}$

(11) If $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ is the coordinate vector of $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ relative to the basis

$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$ of R^3, then $\gamma =$ [8 points]

(a) $\frac{2}{5}$
(b) $\frac{3}{7}$
(c) 1
(d) 0
(e) $\frac{3}{4}$

(12) Let A be an $n \times n$ matrix and $\lambda_1 \neq \lambda_2$ eigenvalues of A. If v_1 is an eigenvector for λ_1 and v_2 is an eigenvector for λ_2, then

[8 points]

(a) v_1 is an eigenvector for λ_1, λ_2.
(b) v_1 and v_2 are linearly independent.
(c) $\lambda_2 v_1$ is not an eigenvector for λ_1.
(d) The matrix A is diagonalizable.
(e) $v_1 + v_2$ is an eigenvector for $\lambda_1 + \lambda_2$.

(13) True or False (circle **T** or **F**) [15 points (3 each)]

a)	Suppose that A is $n \times n$ matrix. We say that an $n \times n$ matrix B is similar to A if there is an invertible matrix P such that $PB = AP$.	**T**	**F**
b)	Any set of vectors containing the zero vector is a linearly dependent set of vectors.	**T**	**F**
c)	The set $\{ x^2, 1+x \}$ span P_2.	**T**	**F**
d)	The set $\{ 1, x, x^2, 2x^2 - 3x + 1 \}$ forms a basis for P_2.	**T**	**F**
e)	The set of all vectors of the form $\begin{bmatrix} 1 \\ x \end{bmatrix}$ is a subspace of R^2.	**T**	**F**