

King Fahd University of Petroleum and Minerals  
 Department of Mathematics and Statistics  
**Math 260 Exam II**  
 Semester I, 2012 (121)

<b>Name:</b>	KEY
<b>ID :</b>	KEY <b>CODE A</b>
<b>Serial no.:</b>	
<b>See</b>	(2: Dr. Al-Shammari) (3: Dr. Al-Humidi) (4: Dr. Al-Humidi) (5: Dr. Mimouni) (6: Dr. Laradji)

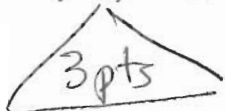
Q		Points
1		12
2		12
3		12
4		12
5		12
6		5
7		5
8		5
9		5
10		5
11		5
12		10 (5 each)
<b>Total</b>		<b>100</b>

(1) Solve the linear differential equation  $xy' + y - x^3 = 0$

[12 points]


putting the equation in the standard form:

$$y' + \frac{1}{x}y = x^2 \quad p(x) = \frac{1}{x} \text{ and } Q(x) = x^2$$



The integrating factor is


$$u = e^{\int p(x) dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$



Then:

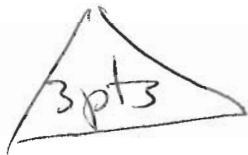
$$y(x) = \frac{1}{u} \left[ \int u Q(x) dx + c \right]$$

$$= \frac{1}{x} \left[ \int x^3 dx + c \right]$$



$$= \frac{1}{x} \left[ \frac{1}{4} x^4 + c \right]$$

$$= \frac{1}{4} x^3 + c \frac{1}{x}$$



(2) Solve the separable differential equation  $(1+x^2)dy + x(1+4y^2)dx = 0$ .

[12 points]

Divide by  $(1+x^2)(1+4y^2)$ ,

$$\frac{dy}{1+4y^2} + \frac{x dx}{1+x^2} = 0$$



$$\Rightarrow \frac{dy}{1+4y^2} = -\frac{x}{1+x^2} dx$$



$$\Rightarrow \int \frac{1}{1+(2y)^2} dy = -\int \frac{x}{1+x^2} dx$$



$$\Rightarrow \frac{1}{2} \tan^{-1}(2y) = -\frac{1}{2} \ln(1+x^2) + C$$



$$\Rightarrow \tan^{-1}(2y) + \ln(1+x^2) = C$$

- (3) Consider the differential equation  $(x^2 + 2x + 3)dx + (x - xy)dy = 0$  (\*) [12 points]

(a) show that the DE is not exact

$$M = x^2 + 2x + 3 \quad , \quad N = x - xy$$

$$M_y = 0 \quad \triangle \quad N_x = 1 - y \quad \triangle$$

$M_y \neq N_x$  so the equation is not exact.

(b) find an integration factor and convert the equation to exact

$$\frac{M_y - N_x}{N} = \frac{0 - (1 - y)}{x(1 - y)} = -\frac{1}{x} \quad \text{depends only on } x \quad \triangle$$

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x} \quad \triangle$$

Multiply the equation (\*) by  $\frac{1}{x}$ , we obtain

(c) solve the DE.

$$\underbrace{\left(-x - 2 - \frac{3}{x}\right)}_{M_2} dx - \underbrace{(1 - y)}_{N_2} dy = 0 \quad \triangle$$

$$(M_2)_y = 0 = (N_2)_x$$

$$F(x, y) = \int M(x, y) dx + g(y) \quad \triangle \text{ 2pts}$$

$$= -\frac{x^2}{2} - 2x - 3 \ln x + g(y)$$

$$\frac{\partial F}{\partial y} = N_2(x, y) \Rightarrow g'(y) = -(1 - y) \quad \text{so } g(y) = \frac{y^2}{2} - y \quad \triangle \text{ 2pts}$$

Thus the solution is given by

$$\boxed{-\frac{x^2}{2} - 2x - 3 \ln x + \frac{1}{2}y^2 - y = C} \quad \triangle \text{ 2pts}$$

- (4) A bacteria culture has an initial population  $P_0$ . After one hour, the number of bacteria is found to be  $\frac{3}{2}P_0$ . If the rate of growth is proportional to the number of bacteria  $P(t)$  present at time  $t$ . Determine the time necessary for the number of bacteria to triple. [12 points]

Let  $P(t)$  be the number of bacteria at time  $t$

then,

$$\frac{dP}{dt} = kP(t) \quad \text{with} \quad \begin{matrix} P(0) = P_0 \\ P(1) = \frac{3}{2}P_0 \end{matrix} \quad \triangle 2$$

We need to find  $t$  s.t.  $P(t) = 3P_0$   $\triangle 1$

Solving the DE:

$$\frac{dP}{P} = k dt \Rightarrow \ln P = kt + C_1 \quad \triangle 2$$

$$\Rightarrow P(t) = Ce^{kt} \quad P(0) = P_0 \Rightarrow C = P_0 \quad \triangle 1$$

$$\Rightarrow P(t) = P_0 e^{kt} \quad \text{Now since } P(1) = \frac{3}{2}P_0$$

$$\Rightarrow P_0 e^k = \frac{3}{2}P_0 \Rightarrow e^k = \frac{3}{2} \Rightarrow k = \ln\left(\frac{3}{2}\right) \quad \triangle 1$$

$$\therefore \boxed{P(t) = P_0 e^{t \ln\left(\frac{3}{2}\right)}} \quad \triangle 3$$

$$P(t) = 3P_0 \Rightarrow 3P_0 = P_0 e^{t \ln\left(\frac{3}{2}\right)} \Rightarrow 3 = e^{t \ln\left(\frac{3}{2}\right)}$$

$$\Rightarrow t \ln\left(\frac{3}{2}\right) = \ln 3 \Rightarrow t = \frac{\ln 3}{\ln\left(\frac{3}{2}\right)} \approx 2.7 \text{ h.}$$

$\triangle 2$

(5) Solve the differential equation  $y^2 = \frac{(y')^3}{y''}$ .

[12 points]

put  $y' = v$   $\triangle 1$  so that  $y'' = v \frac{dv}{dy}$   $\triangle 1$

$$y^2 \cdot y'' = (y')^3$$

$$y^2 v \frac{dv}{dy} = v^3$$

$$(v \neq 0) \triangle 2$$

$$y^2 \frac{dv}{dy} = v^2$$

$$\Rightarrow \frac{dv}{v^2} = \frac{dy}{y^2} \triangle 2$$

$$\Rightarrow \frac{1}{v} = \frac{1}{y} + c_1 \triangle 3$$

So  $dx = \left(\frac{1}{y} + c_1\right) dy$

and then  $x = \ln|y| + c_1 y + c_2 \triangle 3$

(6) The Wronskian of  $f(x) = e^{3x}$  and  $g(x) = e^{-3x}$  is

- (a)  -6
- (b)   $e^{6x}$
- (c)  0
- (d)   $2e^{3x}$
- (e)   $e^{-3x}$

[5 points]

(7)

The matrix  $\begin{bmatrix} 1 & -3 & 2 & 2 \\ 0 & 3 & 6 & 0 \\ 3 & -6 & 0 & 6 \\ 2 & -3 & -2 & 4 \\ -2 & 9 & 2 & -4 \end{bmatrix}$  has rank

- (a)  1
- (b)  2
- (c)  3
- (d)  4
- (e)  5

[5 points]

(8) Let  $P_2$  be the vector space of all polynomials of degree  $\leq 2$ . Which one of the following sets spans  $P_2$ ?

(a)  $\{1, x, x^3\}$

(b)  $\{1, x^2, x^3\}$

(c)  $\{1, x^2, 1+x^2\}$

(d)  $\{1, x, 1+x\}$

(e)  $\{1, 1+x, 1+x^2\}$

[5 points]

(9) Let  $W = \text{Span}\{t, \cos t, \sin t\}$ , a subspace of the space of all continuous functions. Which of the following sets is a basis for  $W$ ?

(a)  $\{t, \cos t, t + \cos t\}$

(b)  $\{t, \sin t, t + \sin t\}$

(c)  $\{\sin t, \cos t, \sin t + \cos t\}$

(d)  $\{t, 2t + 5\sin t, \sin t\}$

(e)  $\{2t, 3\cos t, 5\sin t\}$

[5 points]



(10) The differential equation  $x^2y' + y^2 = x^2y$  is

- (a) Bernoulli  
 (b) Separable  
 (c) Homogeneous  
 (d) Linear  
 (e) Exact

[5 points]

(11) Which of the following statements is **False** for a vector space  $V$  with  $\dim(V) = n$ ?

- (a) Any set of vectors  $\{w_1, w_2, \dots, w_m\}$  with  $m < n$  must be linearly independent.  
 (b) Every set of vectors  $\{w_1, w_2, \dots, w_m\}$  in  $V$  where  $m > n$  is linearly dependent.  
 (c) If  $w_1, w_2, \dots, w_m$  form a basis for  $V$ , then  $m = n$ .  
 (d) If  $w_1, w_2, \dots, w_n$  are linearly independent vectors in  $V$ , then  $w_1, w_2, \dots, w_n$  form basis for  $V$ .  
 (e) If  $\text{Span}\{w_1, w_2, \dots, w_n\} = V$ , then  $w_1, w_2, \dots, w_n$  form a basis for  $V$ .

[5 points]

(12) True or False (circle T or F)

a)	If $A$ is $4 \times 3$ matrix and $\dim(NS(A)) = 1$ , then $\dim(RS(A)) = 2$ .	<input checked="" type="radio"/> T	<input type="radio"/> F
b)	If $f_1, f_2$ are differentiable functions on $(-\infty, \infty)$ and the Wronskian $W(f_1, f_2) = 0$ for all $x \in (-\infty, \infty)$ , then $f_1$ and $f_2$ are linearly dependent.	<input type="radio"/> T	<input checked="" type="radio"/> F

[10 points (5 each)]