Test N°3 Math 232 (December 13, 2012)

KFUPM **Semester 121**

Dept. Math. &Stat. **A.Y:2012/2013**

Name: ....

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Exercise 1

Which of the following functions are one–one, onto, or bijective?

Justify your answer.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x.$
2. $g : \mathbb{N} \to \mathbb{N},\ g(n) = n + 1.$
3. $h : \mathbb{N} \to \mathbb{N}^{+},\ h(n) = n + 1.$
4. $r : \mathbb{R} \to \{0\},\ r(x) = 0.$

Exercise 2

Let S be a set. For each subset $A \subseteq S$ define a function $\chi_A\colon S \to \{0,1\}$, called the **characteristic function** of A, as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

For example, if $S = \{1,2,3,4\}$ and $A = \{1,2,4\}$, then $\chi_A(1) = \chi_A(2) = \chi_A(4) = 1$ and $\chi_A(3) = 0$.

(a) Give the formulas for the characteristic functions $\chi_\emptyset$ and χ_S.

(b) Let A' denote the complement of A in S. Compare the formulas for χ_A and $\chi_{A'}$.

(c) Show that if A and B are subsets of S then

$$A = B \iff \chi_A = \chi_B$$

(The proof will involve the careful consideration of *set equality*, which appears in the left-hand statement, and *function equality*, which appears in the right-hand statement.)

(d) Show that if A and B are subsets of S then

$$\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x) \quad \forall x \in S$$
$$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) \quad \forall x \in S$$

Exercise 3

Let a and b be real numbers. Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = ax + b$.

(a) Under what conditions on a and b is f a bijection from $\mathbb{R}$ to $\mathbb{R}$? (For instance, what if $a = 0$?)

(b) Under what conditions on a and b is the restriction $f|_{\mathbb{Z}}$ a bijection from $\mathbb{Z}$ to $\mathbb{Z}$?

(c) Under what conditions on a and b is the restriction $f|_{\mathbb{N}}$ a bijection from $\mathbb{N}$ to $\mathbb{N}$?

Exercise 4

Suppose $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$ are surjections. Prove that there is a surjection $h\colon \mathbb{N} \to A \cup B$. Suggestion: Consider $f(1), g(1), f(2), g(2), \ldots$

Exercise 5

Suppose $f\colon A \to B$ and $g\colon B \to C$ are functions.

(a) Show that if $g \circ f$ is injective then f is injective.

(b) Show that if $g \circ f$ is surjective then g is surjective.

Exercise 6

In each part of this exercise, give examples of sets A, B, C and functions $f\colon A \to B$ and $g\colon B \to C$ satisfying the indicated properties.

(a) g is not injective but $g \circ f$ *is* injective.

(b) f is not surjective but $g \circ f$ *is* surjective.

(Suggestions: Work with sets having at most three elements; draw pictures.)

Exercise 7

Let $f: A \longrightarrow B$. Prove that, if f is one-to-one, then $f(X) \cap f(Y) = f(X \cap Y)$ for all $X,\ Y \subseteq A$. Is the converse true? Explain.