

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS

S121

MATH 201
FINAL EXAM

NAME:.....MASTER COPY.....ID:.....KEY

Tuesday, January 10, 2013

Location: EXB Center

Time: 8AM- 11AM

MCQ Part

Q	1	2	3	4	5	6	7	8	9	10	
Marks											

Written Part

Q	11	12	13	14	15

Total out of 140:

REMARKS FOR INSTRUCTORS

1) For the MCQ: this is the master key so all (a) are solutions to the corresponding exercises EXCEPT

Exercise #2 solution is $-\pi$ that is (b) in master key (not π)

Exercise # 3

whose solution should be 10 or 6, which is not included in the 5 choices given and so is discarded. Every student will be given full mark for that exercise.

2) Exercise 11 on Lagrange multiplier there was a missprint z instead of 2 which led to little computational problem but the problem is workable and the solution is given.

3) Exercise 13 we consider $0 \leq \theta \leq \frac{\pi}{4}$ otherwise there will be two integrals to evaluate.

PART I: MCQ

1. (7pts) The area enclosed by the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ where $A, B > 0$ is
- (a) πAB
 - (b) AB
 - (c) $\frac{AB}{2}$
 - (d) $2AB$
 - (e) $\frac{\pi}{2}(A^2 + B^2)$

PART I: MCQ (cont')

- 2.(7pts) The slope of the tangent line to the polar curve $r = \frac{1}{\theta}$ at $\theta = \pi$ is
- (a) π
 - (b) $-\pi$
 - (c) $-\frac{1}{\pi^2}$
 - (d) $\frac{1}{\pi}$
 - (e) 0

PART I: MCQ (cont')

3.(7pts) If a sphere with center at $O(3, 8, 1)$ passes through $P(4, 3, -1)$ and $Q(8, Y, 0)$, then Y is equal to

- (a) 4 or 8
- (b) 4
- (c) 12 or 6
- (d) 12
- (e) 6

PART I: MCQ (cont')

4. (7pts) The angle between the vectors $u = \langle 3, 4, -1 \rangle$ and $v = \langle 2, -1, -3 \rangle$ is

- (a) $\cos^{-1} \left(\frac{5}{2\sqrt{91}} \right)$
- (b) $\cos^{-1} \left(\frac{4}{\sqrt{91}} \right)$
- (c) $\cos^{-1} \left(\frac{2}{\sqrt{41}} \right)$
- (d) $\cos^{-1} \left(\frac{3}{\sqrt{51}} \right)$
- (e) $\cos^{-1} \left(\frac{5}{\sqrt{91}} \right)$

PART I: MCQ (cont')

5. (7pts) The volume of the parallelepiped determined by the three vectors $u = \langle 3, 2, 1 \rangle$, $v = \langle 1, 1, 2 \rangle$ and $w = \langle 1, 3, 3 \rangle$ is

- (a) 9
- (b) 13
- (c) 12
- (d) 10
- (e) 8

PART I: MCQ (cont')

6. (7pts) An equation of the plane through $(-1, -2, 3)$ and perpendicular to both the planes $x - 3y + 2z = 7$ and $2x - 2y - z = -3$ is

- (a) $7x + 5y + 4z + 5 = 0$
- (b) $7x + 5y + 3z + 8 = 0$
- (c) $5x + 7y + 4z + 7 = 0$
- (d) $5x + 7y + 3z + 10 = 0$
- (e) $4x + 3y + 7z - 11 = 0$

PART I: MCQ (cont')

7. (7pts) The distance of the point $P(4, 1, -3)$ to the plane that passes through the points $A(1, 1, -1)$, $B(-2, 0, 1)$ and $C(0, 1, 3)$ is

- (a) $\frac{10}{\sqrt{117}}$
- (b) $\frac{6}{\sqrt{119}}$
- (c) $\frac{3}{11}$
- (d) $\frac{13}{\sqrt{71}}$
- (e) $\frac{14}{\sqrt{109}}$

PART I: MCQ (cont')

8 . (7pts) Let

$$f(x, y) = \frac{x^2y}{x^4 + y^2}$$

then $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$ does not exist since

- (a) $\lim_{x \rightarrow 0} f(x, y)$ along the curve $y = \alpha x^2$ depends on α
- (b) $\lim_{x \rightarrow 0} f(x, y)$ along the curve $y = \alpha x^2$ is independent of α
- (c) $\lim_{x \rightarrow 0} f(x, y)$ along the curve $y = x^2$ is equal to $\frac{1}{2}$
- (d) $\lim_{x \rightarrow 0} f(x, y)$ along the curve $y = 0$ is equal to 0
- (e) $\lim_{y \rightarrow 0} f(x, y)$ along the curve $x = 0$ is equal to 0.

PART I: MCQ (cont')

9.(7pts) If $z = f\left(\frac{x}{y}\right)$ where f is a differentiable function, then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is equal to

- (a) $\frac{y-x}{y^2} f'\left(\frac{x}{y}\right)$
- (b) $x f'\left(\frac{x}{y}\right) + \frac{x}{y^2} f'\left(\frac{x}{y}\right)$
- (c) $\frac{x-y}{y^2} f'\left(\frac{x}{y}\right)$
- (d) 0
- (e) $y f'\left(\frac{x}{y}\right) - \frac{x}{y^2} f'\left(\frac{x}{y}\right)$

PART I: MCQ

10. (7pts) The linearization $L(x, y, z)$ of $f(x, y, z) = xyz^2 + y^2z + 2x$ at the point $(1, 0, 1)$ is

- (a) $L(x, y, z) = 2x + y$
- (b) $L(x, y, z) = 2x + y - 2$
- (c) $L(x, y, z) = x + y + 2z + 1$
- (d) $L(x, y, z) = 2x + z + z + 2$
- (e) $L(x, y, z) = x + y + z$

PART II: Written

11. (14pts) Find the minimum and maximum of $f(x, y, z) = -x + 2y + 2z$ subject to the constraints

$$\begin{cases} g(x, y, z) = x^2 + y^2 - z = 0 \\ h(x, y, z) = y + 2z - 1 = 0 \end{cases}$$

Let $F = f - \lambda g - \mu h$ then $\nabla F = 0$ gives the system

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0, \frac{\partial F}{\partial \mu} = 0,$$

that is

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z} \\ g &= 0 \\ h &= 0 \end{aligned}$$

hence,

$$\begin{aligned} -1 &= 2\lambda x \\ 2 &= 2\lambda y + \mu \\ 2 &= -\lambda + 2\mu \\ x^2 + y^2 - z &= 0 \\ y + 2z - 1 &= 0 \end{aligned}$$

from which we get

$$\begin{aligned} x &= \frac{-1}{2\lambda} \\ y &= \frac{2 - \mu}{2\lambda} \\ \lambda &= 2\mu - 2 \end{aligned}$$

Replacing λ in terms of μ in x and y we get

$$\begin{aligned} x &= \frac{-1}{4(\mu - 1)} \\ y &= \frac{2 - \mu}{4(\mu - 1)} \end{aligned}$$

From the constraints equations we have

$$y + 2(x^2 + y^2) = 1$$

giving,

$$\frac{2 - \mu}{4(\mu - 1)} + 2 \left(\frac{1}{16(\mu - 1)^2} + \frac{(2 - \mu)^2}{16(\mu - 1)^2} \right) = 1$$

Reducing to the same denominator and simplifying, we get

$$9\mu^2 - 18\mu + 7 = 0$$

whose solutions are

$$\mu_1 = \frac{1}{3}(3 - \sqrt{2}) \text{ and } \mu_2 = \frac{1}{3}(3 + \sqrt{2}).$$

Hence,

$$\lambda_1 = -\frac{2\sqrt{2}}{3} \text{ and } \lambda_2 = \frac{2\sqrt{2}}{3}$$

and

$$\begin{aligned} x_1 &= \frac{3\sqrt{2}}{8} \text{ and } x_2 = -\frac{3\sqrt{2}}{8} \\ y_1 &= -\frac{1}{8}(2 + 3\sqrt{2}) \text{ and } y_2 = \frac{1}{8}(-2 + 3\sqrt{2}) \\ z_1 &= \frac{1}{16}(10 + 3\sqrt{2}) \text{ and } z_2 = \frac{1}{16}(10 - 3\sqrt{2}) \end{aligned}$$

Evaluation of f at the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ gives $f_1 \simeq -0.31$ and $f_2 \simeq 1.81$ which turn out to be the minimum of f and the maximum of f respectively.

PART II: Written (cont')

12. (14pts) Evaluate

$$I = \int_0^8 \int_{y^{1/3}}^2 e^{x^4} dx dy$$

Since we cannot perform the inner integration, and the integrand is continuous, we shall reverse the order of integrations.
we have,

$$\begin{aligned} I &= \int_0^2 \int_0^{x^3} e^{x^4} dy dx \\ &= \int_0^2 e^{x^4} \int_0^{x^3} dy dx \\ &= \int_0^2 e^{x^4} x^3 dx \\ &= \left(\frac{1}{4} e^{x^4} \right) \Big|_{x=0}^{x=2} \\ &= \frac{1}{4} (e^{16} - 1) \end{aligned}$$

PART II: Written (cont')

13. (14pts) Integrate the function $f(x, y) = xy$ over the region bounded by the four-leaved rose $r = \cos 2\theta$ in the first quadrant ($0 \leq \theta \leq \frac{\pi}{2}$).

Note: HERE WE SHALL CONSIDER $0 \leq \theta \leq \frac{\pi}{4}$ OTHERWISE THERE WILL BE TWO INTEGRALS TO EVALUATE

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r \cos \theta r \sin \theta r dr d\theta \\ &= \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4} \right]_{r=0}^{\cos 2\theta} \cos \theta \sin \theta d\theta \\ &= \frac{1}{8} \int_0^{\frac{\pi}{4}} \cos^4 2\theta \sin 2\theta d\theta \\ &= \left[-\frac{1}{8} \frac{\cos^5 2\theta}{5} \right]_{\theta=0}^{\frac{\pi}{4}} \\ &= \frac{1}{80} \end{aligned}$$

PART II: Written (cont')

14.(14pts) Integrate the function $f(x, y, z) = 6xy$ over the solid E that lies under the plane $z = 1 + x + y$ and above the region in the xy -plane bounded by the curves $y = x^{\frac{1}{2}}$, $y = 0$ and $x = 1$.

A simple sketch of the curve $y = \sqrt{x}$ will show that our integral can be written as

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \cdot dz \cdot dy \cdot dx \\ &= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)dy \cdot dx \\ &= \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2)dy \cdot dx \\ &= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})dx \\ &= (x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2})_{x=0}^{x=1} \\ &= 1 + \frac{3}{4} + \frac{4}{7} \\ &= \frac{65}{28} \end{aligned}$$

PART II: Written (cont')

15. (14pts) In the spherical coordinate system, find the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4 \cos \phi$.

The solid lies in

$$E = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 4 \cos \phi, 0 \leq \phi \leq \frac{\pi}{3}, 0 \leq \theta \leq 2\pi\}$$

Thus,

$$\begin{aligned} V &= \int \int \int_E dv \\ &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4 \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \frac{64}{3} \cos^3 \phi \sin \phi d\phi d\theta \\ &= \frac{128\pi}{3} \int_0^{\pi/3} \cos^3 \phi \sin \phi d\phi \\ &= \frac{128\pi}{3} \left[-\frac{\cos^4 \phi}{4} \right]_0^{\pi/3} \\ &= \frac{128\pi}{12} \left[1 - \frac{1}{16} \right] \\ &= 10\pi \end{aligned}$$