

Exercise #1: (10 pts) Find the distance from the point $D(1, 1, -3)$ to the plane containing the points $A(1, 0, -2)$, $B(0, -1, 2)$, $C(1, 1, 1)$.

Solution:

Equation of the plane containing the points A, B, C :

$$(2) \quad \vec{AB} = \langle -1, -1, 4 \rangle, \quad \vec{AC} = \langle 0, 1, 3 \rangle$$

Normal vector:

$$\vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & -1 & 4 \\ 0 & 1 & 3 \end{vmatrix}$$

$$(2) \quad = -7i + 3j - k$$

Equation of plane ABC:

$$(3) \quad -7(x-1) + 3(y-0) - (z+2) = 0$$

that is $-7x + 3y - z + 5 = 0$

Distance from $D(1, 1, -3)$ to the plane ABC:

$$(3) \quad d = \frac{|-7(1) + 3(1) - (-3) + 5|}{\sqrt{(-7)^2 + (3)^2 + (-1)^2}} = \frac{4}{\sqrt{59}}$$

Exercise #2: (11 pts) Find the area of the triangle whose vertices are P(1, 3, 2), Q(4, 8, 1) and R(2, 2, 3).

Solution:

we have $\vec{PQ} = (4-1)i + (8-3)j + (1-2)k$ (1)
 $= 3i - 5j - k$

$$\vec{PR} = (2-1)i + (2-3)j + (3-2)k$$
$$= i - j + k$$
 (1)

so that

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 3 & -5 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$
 (2)

$$= (5-1)i - (3-(-1))j + (-3-5)k$$
$$= 4i - 4j - 8k.$$
 (2)

the area of the triangle is therefore,

$$A = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$$
 (2)

$$= \frac{1}{2} \sqrt{(4)^2 + (-4)^2 + (-8)^2}$$
 (1)

$$= \frac{1}{2} \sqrt{16 + 16 + 64}$$

$$= \frac{1}{2} \sqrt{96}$$

$$= 2\sqrt{6} \quad \text{unit}^2$$
 (2)

Exercise #3:(11 pts) Find an equation of the ellipsoid whose center is at $(1, 1, 1)$ and which passes through the points $(0, 0, 1)$, $(1, 1, \sqrt{5} + 1)$ and $(1, 0, 3)$.

Solution:

An equation of the ellipsoid is given by

$$a(x-1)^2 + b(y-1)^2 + c(z-1)^2 = 1 \quad (4)$$

since it passes through $(0, 0, 1)$ we have,

$$a + b = 1 \quad (1)$$

since it passes through $(1, 1, \sqrt{5} + 1)$ we have,

$$5c = 1 \quad (1)$$

since it passes through $(1, 0, 3)$ we have,

$$b + 4c = 1 \quad (1)$$

Thus, a, b, c must satisfy the system,

$$\begin{cases} a + b = 1 \\ 5c = 1 \\ b + 4c = 1 \end{cases} \quad (1)$$

whose solution is $a = 4/5$, $b = 1/5$, $c = 1/5$ (2)

Therefore an equation of the ellipsoid is

$$\frac{4}{5}(x-1)^2 + \frac{1}{5}(y-1)^2 + \frac{1}{5}(z-1)^2 = 1. \quad (1)$$

Exercise #4: (15 pts) Consider the function

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2} \ln(z - x^2 - y^2)$$

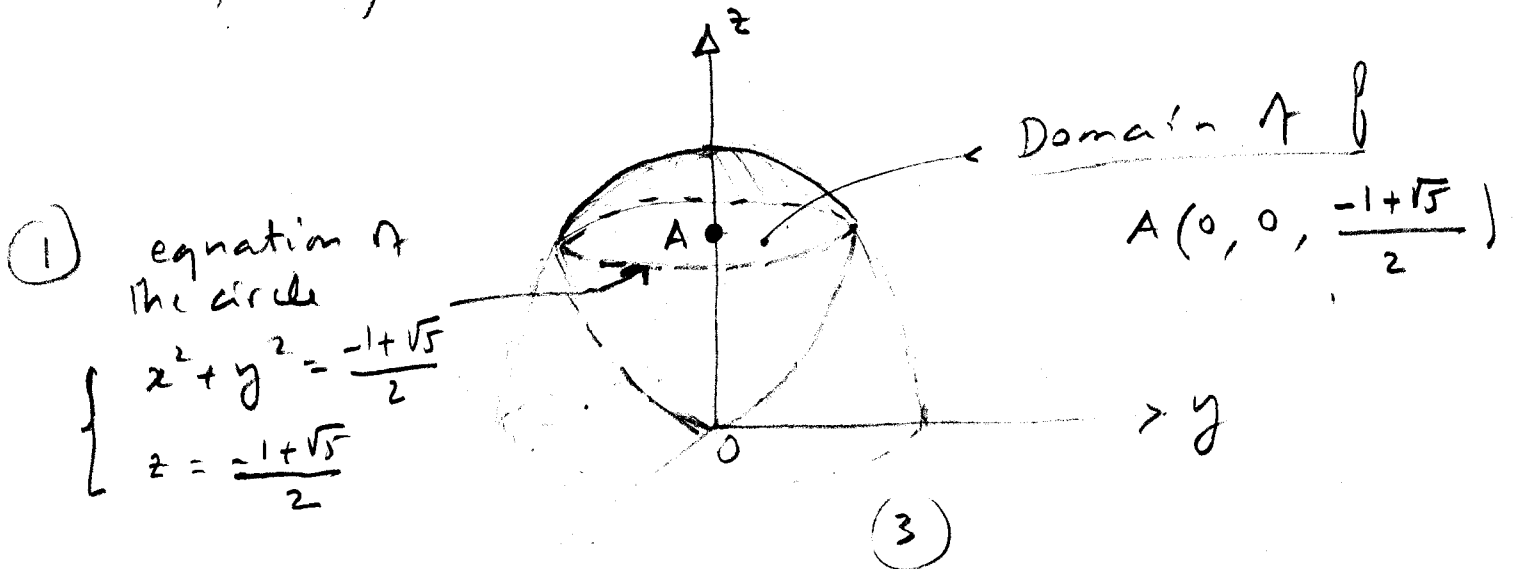
(a) Find the domain of f

We must have $z - x^2 - y^2 > 0$ and $1 - x^2 - y^2 - z^2 \geq 0$ which give the domain:

$$\text{Domain of } f = \{(x, y, z) \in \mathbb{R}^3 : z > x^2 + y^2 \text{ and } x^2 + y^2 + z^2 \leq 1\}$$

(b) Identify and sketch the domain of f .

The domain of f consists of all points inside the paraboloid $z = x^2 + y^2$ and inside or on the sphere $x^2 + y^2 + z^2 = 1$; that is



A was obtained by solving $z + z^2 = 1$.
 points on the paraboloid are excluded
 points on the sphere are included

(1)
(1)
(1)

Exercise #6: (14 pts)

(a) Show that $f(x, y) = \frac{x}{x+y}$ is differentiable at the point $(2, 1)$.

① $f_x(x, y) = \frac{(x+y) - x}{(x+y)^2} = \frac{y}{(x+y)^2}$ ① $f_y(x, y) = \frac{-x}{(x+y)^2}$

① f is continuous and has continuous partial derivatives ① at $(2, 1)$ and in a small neighborhood of $(2, 1)$. Thus, f is differentiable at $(2, 1)$.
①

(b) Find the linearization $L(x, y, z)$ of $f(x, y, z) = \tan^{-1}(xyz)$ at the point $(1, 1, 1)$.

① f is differentiable at $(1, 1, 1)$ as composition of differentiable functions.

① we have $f(1, 1, 1) = \tan^{-1} 1 = \frac{\pi}{4}$.

① $f_x(x, y, z) = \frac{y^2 z}{1 + x^2 y^2 z^2} \Rightarrow f_x(1, 1, 1) = \frac{1}{2}$

① $f_y(x, y, z) = \frac{x^2 z}{1 + x^2 y^2 z^2} \Rightarrow f_y(1, 1, 1) = \frac{1}{2}$

① $f_z(x, y, z) = \frac{xy}{1 + x^2 y^2 z^2} \Rightarrow f_z(1, 1, 1) = \frac{1}{2}$

Thus, the linearization

③ $L(x, y, z) = f(1, 1, 1) + f_x(1, 1, 1)(x-1) + f_y(1, 1, 1)(y-1) + f_z(1, 1, 1)(z-1)$

of f at $(1, 1, 1)$ is

① $L(x, y, z) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}(z-1) = \frac{\pi}{4} - \frac{3}{2} + \frac{x+y+z}{2}$

Exercise #5: (14 pts) Find the following limits if they exist,

(a)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2}$$

(2) Using polar coordinates we have $x = r \cos \theta$,
 $y = r \sin \theta$ so that

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2-y^2} - 1}{x^2 + y^2} &= \lim_{r \rightarrow 0} \frac{e^{-r^2} - 1}{r^2} = \lim_{\alpha \rightarrow 0} \frac{e^{-\alpha} - 1}{\alpha} \\ &= \lim_{\alpha \rightarrow 0} \frac{(e^{-\alpha} - 1)'}{(\alpha)'} = \lim_{\alpha \rightarrow 0} \frac{-e^{-\alpha}}{1} = -1. \end{aligned}$$

(b)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6}$$

Along the path $y = 0$ we have

$$(2) \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} \frac{xy^3}{x^2 + y^6} = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0$$

Along the path $x = y^3$ we have

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6} = \lim_{y \rightarrow 0} \frac{y^6}{2y^6} = \lim_{y \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

(1) These two limits are different so

$$(2) \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^6} \text{ does not exist.}$$

(other paths are possible)

Exercise #7: (11 pts) The temperature of a body at a point (x, y, z) is given by

$$T = e^{xy} - xy^2 - x^2yz.$$

Determine the direction of the greatest drop in the temperature at the point $(1, -1, 2)$.

Solution: The greatest drop (decrease) at the point $(1, -1, 2)$ is in the direction of the negative gradient at that point.

$$\nabla T = (y e^{xy} - y^2 - 2xyz) \mathbf{i} + (x e^{xy} - 2xy - x^2z) \mathbf{j} + (-x^2y) \mathbf{k}$$

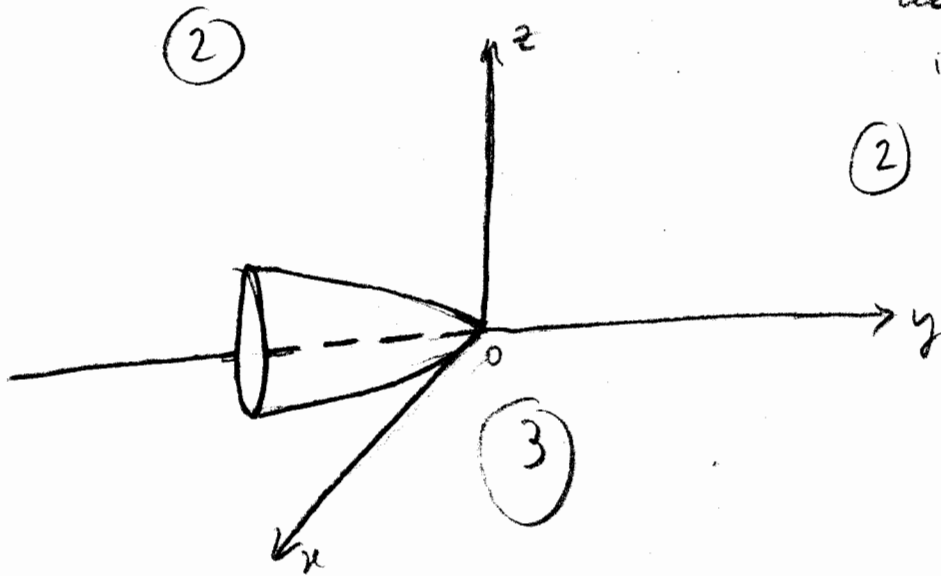
$-\nabla T(1, -1, 2) = \langle e^{-1} - 3, -e^{-1}, -1 \rangle$
is the direction of the greatest drop in the temperature at the point $(1, -1, 2)$.

Exercise #8 (14 pts) Identify the following surfaces and sketch each one in 3D-space,

(a) $x^2 + 4y + 9z^2 = 0$

$$-y = \frac{1}{4}x^2 + \frac{9}{4}z^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{z}{2/3}\right)^2$$

Elliptic paraboloid
along the y-axis
in the direction
 $y < 0$.



(b) $x^2 - 4y^2 + z^2 - 6x - 8y - 2z + 6 = 0$

$$x^2 - 6x - 4y^2 - 8y + z^2 - 2z + 6 = 0$$

$$[(x-3)^2 - 9] - 4(y+1)^2 + 4 + (z-1)^2 - 1 + 6 = 0$$

$$(x-3)^2 - \left(\frac{y+1}{\frac{1}{2}}\right)^2 + (z-1)^2 = 0$$

$$\left(\frac{y+1}{\frac{1}{2}}\right)^2 = (x-3)^2 + (z-1)^2$$

Circular cone with vertex at $(3, -1, 1)$
and axis along the y-axis

