Serial No.:______ Student Name: Key Student Number: ______

Instructor: M. Z. Abu-Sbeih Math 201.11 – Q2 Date: 3-10-2012

Show all your work. No credits for answers not supported by work.

Write neatly and eligibly. You may loose credits for messy work.

Problem 1: (20 Points) (a) Prove that the three points $A(1,3,5), B(-2,0,3), C(7,9,9)$ are collinear.

$\vec{AB} = \langle -3, -3, -2\rangle$ $\vec{AC} = \langle 6, 6, 4\rangle = -2\langle -3, -3, -2\rangle$.

$\therefore$ two vectors are $\parallel$. Since they have a common poin A, they coinside.

(b) Prove that the triangle with vertices $A(2,0,4\sqrt{2}), B(3,-1,5\sqrt{2})$ and $C(4,-2,4\sqrt{2})$ is an isosceles right –angle triangle.

$$|\vec{AB}|^2 = (3-2)^2 + (-1-0)^2 + (5\sqrt{2}-4\sqrt{2})^2 = 4$$
$$|\vec{BC}|^2 = (4-3)^2 + (-2+1)^2 + (4\sqrt{2}-5\sqrt{2})^2 = 4$$
$\Big\}\ |\vec{AB}| = |\vec{BC}|$.
$$|\vec{AC}|^2 = (4-2)^2 + (-2-0)^2 + (4\sqrt{2}-4\sqrt{2})^2 = 8$$

Also $|\vec{AB}|^2 + |\vec{BC}|^2 = |\vec{AC}|^2 \Rightarrow$ angle $\hat{B} = \frac{\pi}{2}$

(c) Find the equation of a sphere with center (1,2,3) and touches the yz-plane.

Radius: $r = x$ – coordinate

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$$

(d) Describe geometrically the graph of the equation

$$x^2 + y^2 + z^2 + 2x - 4y + 5 = 0$$

Complete the squares

$$(x^2+2x+1) + (y^2-4y+4) + z^2 = 0$$
$$(x+1)^2 + (y-2)^2 + z^2 = 0.$$

The graph is one point $(-1, 2, 0)$.

Problem 2: (24 Points) Let $A(1,2,0), B(2,1,1), C(1,1,2)$ be three points in space.

a) Find a unit vector in the direction of $\vec{u} = \vec{AB}$. $= \langle 2-1, 1-2, 1-0\rangle = \langle 1, -1, 1\rangle$.

unit vector in the direction of $\vec{u}$: $\frac{\langle 1,-1,1\rangle}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}\langle 1, -1, 1\rangle$.

b) Find $\vec{AB} - 2\,\vec{AC}$.

Note: $\vec{AC} = \langle 0, -1, 2\rangle$.

$$\Rightarrow \vec{AB} - 2\vec{AC} = \langle 1,-1,1\rangle - 2\langle 0,-1,2\rangle$$
$$= \langle 1, 1, -3\rangle$$

c) Find a vector $\vec{w}$ that has opposite direction to $\vec{u}$ and has length 6.

$$\vec{w} = \frac{-6\vec{u}}{|u|} = \frac{-6}{\sqrt{3}}\langle 1,-1,1\rangle = -2\sqrt{3}\langle 1,-1,1\rangle$$

d) Find the scalar projection of $\vec{u}$ onto $\vec{v} = \overrightarrow{AC}$.

$\vec{u} = \langle 1,-1,1 \rangle$; $\vec{v} = \langle 0,-1,2 \rangle$.

$\text{Comp}_{\vec{v}} \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{3}{\sqrt{5}}$

e) Find the vector projection of $\vec{u}$ onto $\vec{v} = \overrightarrow{AC}$.

$$\text{Proj}_{\vec{v}} \vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot \vec{v} = \frac{3}{(\sqrt{5})^2}\cdot\langle 0,-1,2\rangle$$

$$= \frac{3}{5}\langle 0,-1,2\rangle.$$

f) Find the orthogonal projection of $\vec{u}$ onto $\vec{v} = \overrightarrow{AC}$.

$$\text{orth}_{\vec{v}} \vec{u} = \vec{u} - \text{Proj}_{\vec{v}} \vec{u} = \langle 1,-1,1\rangle - \frac{3}{5}\langle 0,-1,2\rangle.$$

$$= \left\langle 1, -\frac{2}{5}, -\frac{1}{5}\right\rangle$$

<u>Problem 2:</u> (16 Points) (a) Find the angle between the vectors $\vec{u} = \langle 1,-1,0\rangle$ and $\vec{v} = \langle 2,-1,1\rangle$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{3}{\sqrt{2}\sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}.$$

(b) Determine whether the two vectors $\vec{u} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\vec{v} = \langle 1,1,-1\rangle$ are orthogonal, parallel or neither.

Note that $u\cdot v = 2(1) - 1(1) + 1(-1) = 0$

$\Rightarrow$ vectors are orthogonal.

(c) Two direction angles of a vector are $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
$$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1.$$
$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$$
$$\cos^2\gamma = 1 - \frac{3}{4} = \frac{1}{4}$$
$$\Rightarrow \cos\gamma = \frac{1}{2} \Rightarrow \boxed{\gamma = \frac{\pi}{3}} \qquad \cos\gamma = -\frac{1}{2} \Rightarrow \boxed{\gamma = \frac{2\pi}{3}}$$

(d) Find the angle between the diagonal of a cube and the diagonal of one of its faces.

We may assume that the length of cube is 1.

The diagonal of cube: $\vec{b} = \overrightarrow{OP} = \langle 1,1,1\rangle$

The " " face: $\vec{a} = \overrightarrow{QO} = \langle 1,1,0\rangle$.

θ = angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{a\cdot b}{|a||b|} = \frac{2}{\sqrt{1+1+1}\,\sqrt{1+1+0}} = \frac{2}{\sqrt{6}}$$