KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS

S121

MATH 201
FINAL EXAM

NAME:.......MASTER COPY...........ID:......KEY
Tuesday, January 10, 2013
Location: EXB Center
Time: 8AM– 11AM

MCQ Part

Q	1	2	3	4	5	6	7	8	9	10	
Marks											

Written Part

Q	11	12	13	14	15

Total out of 140:

REMARKS FOR INSTRUCTORS

1) **For the MCQ: this is the master key so all (a) are solutions to the corresponding exercises EXCEPT**

Exercise #2 solution is $-\pi$ that is (b) in master key (not π)

Exercise # 3

whose solution should be 10 or 6, which is not included in the 5 choices given and so is discarded. Every student will be given full mark for that exercise.

2) Exercise 11 on Lagrange multiplier there was a missprint z instead of 2 which led to little computational problem but the problem is workable and the solution is given.

3) Exercise 13 we consider $0 \leq \theta \leq \frac{\pi}{4}$ otherwise there will be two integrals to evaluate.

PART I: MCQ

1. (7pts) The area enclosed by the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ where $A, B > 0$ is
(a) πAB
(b) AB
(c) $\frac{AB}{2}$
(d) $2AB$
(e) $\frac{\pi}{2}(A^2 + B^2)$

PART I: MCQ (cont')

2.(7pts) The slope of the tangent line to the polar curve $r = \frac{1}{\theta}$ at $\theta = \pi$ is
(a) π
(b) $-\pi$
(c) $-\frac{1}{\pi^2}$
(d) $\frac{1}{\pi}$
(e) 0

PART I: MCQ (cont')

3.(7pts) If a sphere with center at O$(3,8,1)$ passes through P$(4,3,-1)$and Q$(8,Y,0)$, then Y is equal to

(a) 4 or 8
(b) 4
(c) 12 or 6
(d) 12
(e) 6

PART I: MCQ (cont')

4. (7pts) The angle between the vectors $u =< 3,4,-1 >$ and $v =< 2,-1,-3 >$ is

(a) $\cos^{-1}\left(\frac{5}{2\sqrt{91}}\right)$
(b) $\cos^{-1}\left(\frac{4}{\sqrt{91}}\right)$
(c) $\cos^{-1}\left(\frac{2}{\sqrt{41}}\right)$
(d) $\cos^{-1}\left(\frac{3}{\sqrt{51}}\right)$
(e) $\cos^{-1}\left(\frac{5}{\sqrt{91}}\right)$

PART I: MCQ (cont')

5. (7pts) The volume of the parallelepipede determined by the three vectors $u =< 3, 2, 1 >$, $v =< 1, 1, 2 >$ and $w =< 1, 3, 3 >$ is

(a) 9
(b) 13
(c) 12
(d) 10
(e) 8

PART I: MCQ (cont')

6. (7pts) An equation of the plane through $(-1, -2, 3)$ and perpendicular to both the planes $x - 3y + 2z = 7$ and $2x - 2y - z = -3$ is

(a) $7x + 5y + 4z + 5 = 0$
(b) $7x + 5y + 3z + 8 = 0$
(c) $5x + 7y + 4z + 7 = 0$
(d) $5x + 7y + 3z + 10 = 0$
(e) $4x + 3y + 7z - 11 = 0$

PART I: MCQ (cont')

7. (7pts) The distance of the point P$(4, 1, -3)$ to the plane that passes through the points A$(1, 1, -1)$, B$(-2, 0, 1)$ and C$(0, 1, 3)$ is

(a) $\frac{10}{\sqrt{117}}$
(b) $\frac{6}{\sqrt{119}}$
(c) $\frac{3}{11}$
(d) $\frac{13}{\sqrt{71}}$
(e) $\frac{14}{\sqrt{109}}$

PART I: MCQ (cont')

8 . (7pts) Let

$$f(x, y) = \frac{x^2 y}{x^4 + y^2}$$

then $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist since

(a) $\lim_{x \to 0} f(x, y)$ along the curve $y = \alpha x^2$ depends on α
(b) $\lim_{x \to 0} f(x, y)$ along the curve $y = \alpha x^2$ is independent of α
(c) $\lim_{x \to 0} f(x, y)$ along the curve $y = x^2$ is equal to $\frac{1}{2}$
(d) $\lim_{x \to 0} f(x, y)$ along the curve $y = 0$ is equal to 0
(e) $\lim_{y \to 0} f(x, y)$ along the curve $x = 0$ is equal to 0.

PART I: MCQ (cont')

9.(7pts) If $z = f(\frac{x}{y})$ where f is a differentiable function, then $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is equal to

(a) $\frac{y-x}{y^2} f'(\frac{x}{y})$
(b) $x f'(\frac{x}{y}) + \frac{x}{y^2} f'(\frac{x}{y})$
(c) $\frac{x-y}{y^2} f'(\frac{x}{y})$
(d) 0
(e) $y f'(\frac{x}{y}) - \frac{x}{y^2} f'(\frac{x}{y})$

PART I: MCQ

10. (7pts) The linearization $L(x, y, z)$ of $f(x, y.z) = xyz^2 + y^2 z + 2x$ at the point $(1, 0, 1)$ is

(a) $L(x, y, z) = 2x + y$
(b) $L(x, y, z) = 2x + y - 2$
(c) $L(x, y, z) = x + y + 2z + 1$
(d) $L(x, y, z) = 2x + z + z + 2$
(e) $L(x, y, z) = x + y + z$

PART II: Written

11. (14pts) Find the minimum and maximum of $f(x,y,z) = -x+2y+2z$ subject to the constraints

$$\begin{cases} g(x,y,z) = x^2+y^2-z=0 \\ h(x,y,z) = y+2z-1=0 \end{cases}$$

Let $F = f - \lambda g - \mu h$ then $\nabla F = 0$ gives the system

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = 0, \frac{\partial F}{\partial \mu} = 0,$$

that is

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda\frac{\partial g}{\partial x} + \mu\frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda\frac{\partial g}{\partial y} + \mu\frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda\frac{\partial g}{\partial z} + \mu\frac{\partial h}{\partial z} \\ g &= 0 \\ h &= 0 \end{aligned}$$

hence,

$$\begin{aligned} -1 &= 2\lambda x \\ 2 &= 2\lambda y + \mu \\ 2 &= -\lambda + 2\mu \\ x^2+y^2-z &= 0 \\ y+2z-1 &= 0 \end{aligned}$$

from which we get

$$\begin{aligned} x &= \frac{-1}{2\lambda} \\ y &= \frac{2-\mu}{2\lambda} \\ \lambda &= 2\mu - 2 \end{aligned}$$

Replacing λ in terms of μ in x and y we get

$$\begin{aligned} x &= \frac{-1}{4(\mu-1)} \\ y &= \frac{2-\mu}{4(\mu-1)} \end{aligned}$$

From the constraints equations we have

$$y + 2(x^2 + y^2) = 1$$

giving,

$$\frac{2-\mu}{4(\mu-1)} + 2\left(\frac{1}{16(\mu-1)^2} + \frac{(2-\mu)^2}{16(\mu-1)^2}\right) = 1$$

Reducing to the same denominator and simplifying, we get

$$9\mu^2 - 18\mu + 7 = 0$$

whose solutions are

$$\mu_1 = \frac{1}{3}(3-\sqrt{2}) \text{ and } \mu_2 = \frac{1}{3}(3+\sqrt{2}).$$

Hence,

$$\lambda_1 = -\frac{2\sqrt{2}}{3} \text{ and } \lambda_2 = \frac{2\sqrt{2}}{3}$$

and

$$\begin{aligned} x_1 &= \frac{3\sqrt{2}}{8} \text{ and } x_2 = -\frac{3\sqrt{2}}{8} \\ y_1 &= -\frac{1}{8}(2+3\sqrt{2}) \text{ and } y_2 = \frac{1}{8}(-2+3\sqrt{2}) \\ z_1 &= \frac{1}{16}(10+3\sqrt{2}) \text{ and } z_2 = \frac{1}{16}(10-3\sqrt{2}) \end{aligned}$$

Evaluation of f at the points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ gives $f_1 \simeq -0.31$ and $f_2 \simeq 1.81$ which turn out to be the minimum of f and the maximum of f respectively.

PART II: Written (cont')

12. (14pts) Evaluate

$$I = \int_0^8 \int_{y^{1/3}}^2 e^{x^4}\, dx dy$$

Since we cannot perform the inner integration, and the integrand is continuous, we shall reverse the order of integrations.

we have,

$$\begin{aligned} I &= \int_0^2 \int_0^{x^3} e^{x^4} dy dx \\ &= \int_0^2 e^{x^4} \int_0^{x^3} dy dx \\ &= \int_0^2 e^{x^4} x^3 dx \\ &= \left(\frac{1}{4} e^{x^4}\right)_{|x=0}^{x=2} \\ &= \frac{1}{4}(e^{16} - 1) \end{aligned}$$

13. (14pts) Integrate the function $f(x, y) = xy$ over the region bounded by the four-leaved rose $r = \cos 2\theta$ in the first quadrant $(0 \leq \theta \leq \frac{\pi}{2})$.

Note: HERE WE SHALL CONSIDER $0 \leq \theta \leq \frac{\pi}{4}$OTHERWISE THERE WILL BE TWO INTEGRALS TO EVALUATE

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \int_o^{\cos 2\theta} r \cos\theta r \sin\theta r dr d\theta \\
&= \int_0^{\frac{\pi}{4}} \left[\frac{r^4}{4}\right]_{r=0}^{\cos 2\theta} \cos\theta \sin\theta d\theta \\
&= \frac{1}{8} \int_0^{\frac{\pi}{4}} \cos^4 2\theta \sin 2\theta d\theta \\
&= \left[-\frac{1}{8}\frac{\cos^5 2\theta}{10}\right]_{\theta=0}^{\frac{\pi}{4}} \\
&= \frac{1}{80}
\end{aligned}
$$

PART II: Written (cont')

14.(14pts) Integrate the function $f(x, y, z) = 6xy$ over the solid E that lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = x^{\frac{1}{2}}$, $y = 0$ and $x = 1$.

A simple sketch of the curve $y = \sqrt{x}$ will show that our integral can be written as

$$\begin{aligned}
I &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \cdot dz \cdot dy \cdot dx \\
&= \int_0^1 \int_0^{\sqrt{x}} 6xy(1 + x + y)dy \cdot dx \\
&= \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2)dy \cdot dx \\
&= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})dx \\
&= (x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2})_{x=0}^{x=1} \\
&= 1 + \frac{3}{4} + \frac{4}{7} \\
&= \frac{65}{28}
\end{aligned}$$

PART II: Written (cont')

15. (14pts) In the spherical coordinate system, find the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4\cos\phi$.

The solide lies in

$$E = \{(\rho, \phi, \theta) :\} \, 0 \le \rho \le 4\cos\phi, 0 \le \phi \le \frac{\pi}{3}, 0 \le \theta \le 2\pi$$

Thus,

$$\begin{aligned}
V &= \int\int\int_E dv \\
&= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \frac{64}{3}\cos^3\phi \sin\phi d\phi d\theta \\
&= \frac{128\pi}{3}\int_0^{\pi/3} \cos^3\phi \sin\phi d\phi \\
&= \frac{128\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{3}} \\
&= \frac{128\pi}{12}\left[1 - \frac{1}{16}\right] \\
&= 10\pi
\end{aligned}$$