KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS

S121

MATH 201
MAJOR EXAM II
Thursday, November 22 , 2012
Time: 5:30 PM - 7:30 PM
Location: Bldg 54

NAME:..............................ID: Key ....SECTION.........

Instructions: Formula sheets, calculators and mobiles are not allowed.

JUSTIFY ALL YOURS ANSWERS

Question #	1	2	3	4	5	6	7	8	Total
Points	10	11	11	15	14	14	11	14	100
Marks									

Exercise #1:(10 pts) Find the distance from the point D(1, 1, −3) to the plane containing the points A(1, 0, −2), B(0, −1, 2), C(1, 1, 1).

Solution :

Equation of the plane containing the points A, B, C :

(2) $\overrightarrow{AB} = \langle -1, -1, 4 \rangle$, $\overrightarrow{AC} = \langle 0, 1, 3 \rangle$

Normal vector :

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & -1 & 4 \\ 0 & 1 & 3 \end{vmatrix}$$

(2) $= -7i + 3j - k$.

Equation of plane ABC :

(3) $-7(x-1) + 3(y-0) - (z+2) = 0$

that is $-7x + 3y - z + 5 = 0$

Distance from D(1, 1, −3) to the plane ABC:

(3)
$$d = \frac{|-7(1)+3(1)-(-3)+5|}{\sqrt{(-7)^2+(3)^2+(-1)^2}} = \frac{4}{\sqrt{59}}$$

Exercise #2:(11 pts) Find the area of the triangle whose vertices are P(1, 3, 2), Q(4, 8, 1) and R(2, 2, 3).

solution:

we have

$$\vec{PQ} = (4-1)i + (8-3)j + (1-2)k \qquad (1)$$
$$= 3i - 5j - k$$

$$\vec{PR} = (2-1)i + (2-3)j + (3-2)k \qquad (1)$$
$$= i - j + k$$

so that

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 3 & 5 & -1 \\ 1 & -1 & 1 \end{vmatrix} \qquad (2)$$

$$= (5-1)i - (3-(-1))j + (-3-5)k$$
$$= 4i - 4j - 8k. \qquad (2)$$

the area of the triangle is therefore,

$$A = \frac{1}{2}|\vec{PQ} \times \vec{PR}| \qquad (2)$$
$$= \frac{1}{2}\sqrt{(4)^2 + (-4)^2 + (-8)^2} \qquad (1)$$
$$= \frac{1}{2}\sqrt{16 + 16 + 64}$$
$$= \frac{1}{2}\sqrt{96} \qquad (2)$$
$$= 2\sqrt{6} \quad \text{unit}^2$$

Exercise #3:(11 pts) Find an equation of the ellipsoid whose center is at $(1,1,1)$ and which passes through the points $(0,0,1)$,$(1,1,\sqrt{5}+1)$ and $(1,0,3)$.

Solution:

An equation of the ellipsoid is given by

$$a(x-1)^2 + b(y-1)^2 + c(z-1)^2 = 1$$ 4

since it passes through $(0,0,1)$ we have,

$$a + b = 1$$ 1

since it passes through $(1,1,\sqrt{5}+1)$ we have,

$$5c = 1$$ 1

since it passes through $(1,0,3)$ we have,

$$b + 4c = 1$$ 1

Thus, a, b, c must satisfy the system,

$$\begin{cases} a + b = 1 \\ 5c = 1 \\ b + 4c = 1 \end{cases}$$ 1

whose solution is $a = 4/5$, $b = 1/5$, $c = 1/5$ 2

Therefore an equation of the ellipsoid is

$$\frac{4}{5}(x-1)^2 + \frac{1}{5}(y-1)^2 + \frac{1}{5}(z-1)^2 = 1.$$ 1

Exercise #4:(15 pts) Consider the function

$$f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}\,\ln(z - x^2 - y^2)$$

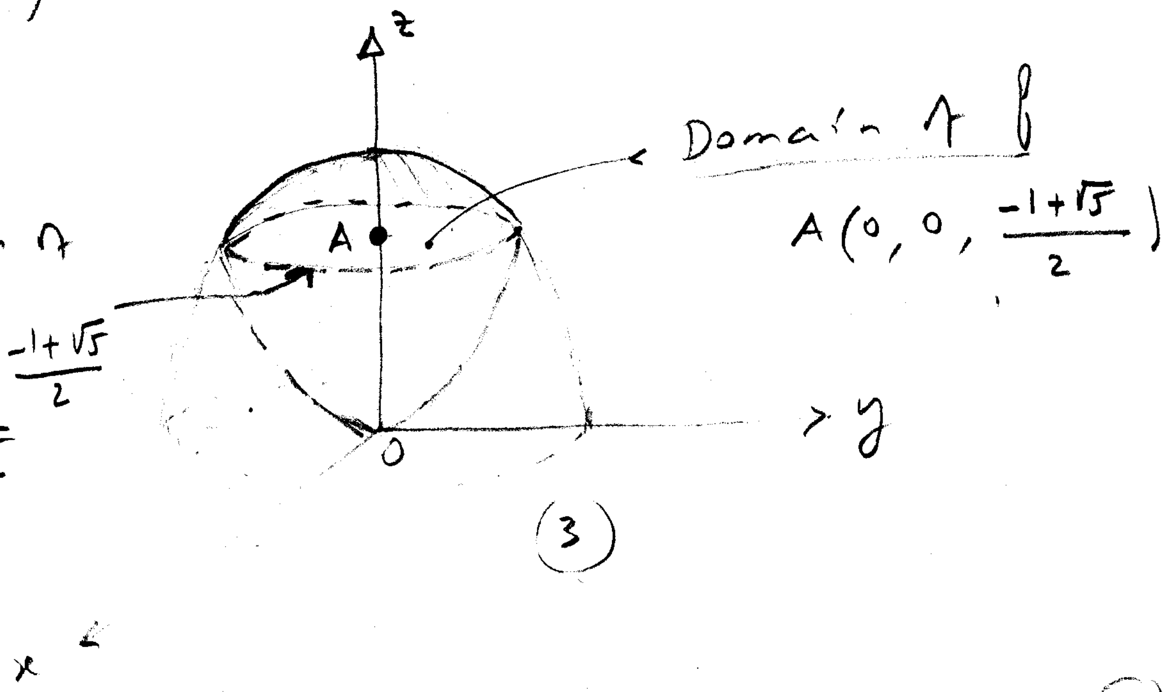
(a) Find the domain of f 2 2

We must have $z - x^2 - y^2 > 0$ and $1 - x^2 - y^2 - z^2 \geq 0$
which give the domain:

Domain of $f = \{(x,y,z) \in \mathbb{R}^3 : z > x^2 + y^2$ and $x^2 + y^2 + z^2 \leq 1\}$ 2

(b) Identify and sketch the domain of f.

2 The domain of f consists of all points inside the paraboloid $z = x^2 + y^2$ and inside or on the sphere $x^2 + y^2 + z^2 = 1$; that is



1 equation of the circle

$$\begin{cases} x^2 + y^2 = \frac{-1+\sqrt{5}}{2} \\ z = \frac{-1+\sqrt{5}}{2} \end{cases}$$

A was obtained by solving $z + z^2 = 1$. 1
points on the paraboloid are excluded 1
point on the sphere are included 1

Exercise #6:(14 pts)

(a) Show that $f(x,y) = \frac{x}{x+y}$ is differentiable at the point $(2,1)$.

(1) $f_x(x,y) = \dfrac{(x+y)-x}{(x+y)^2} = \dfrac{y}{(x+y)^2}$ (1) $f_y(x,y) = \dfrac{-x}{(x+y)^2}$

(1) f is continuous and has continuous partial derivatives (1) at $(2,1)$ and in a small neighborhood of $(2,1)$. Then, f is differentiable at $(2,1)$.

(1)

(b) Find the linearization $L(x,y,z)$ of $f(x,y,z) = \tan^{-1}(xyz)$ at the point $(1,1,1)$.

(1) f is differentiable at $(1,1,1)$ as composition of differentiable functions.

(1) We have $f(1,1,1) = \tan^{-1} 1 = \frac{\pi}{4}$.

(1) $f_x(x,y,z) = \dfrac{yz}{1+x^2y^2z^2} \Rightarrow f_x(1,1,1) = \frac{1}{2}$

(1) $f_y(x,y,z) = \dfrac{xz}{1+x^2y^2z^2} \Rightarrow f_y(1,1,1) = \frac{1}{2}$

(1) $f_z(x,y,z) = \dfrac{xy}{1+x^2y^2z^2} \Rightarrow f_z(1,1,1) = \frac{1}{2}$

Thus, the linearization

(3) $L(x,y,z) = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$

of f at $(1,1,1)$ is

(1) $L(x,y,z) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}(z-1) = \frac{\pi}{4} - \frac{3}{2} + \frac{x+y+z}{2}$

Exercise #5:(14 pts) Find the following limits if they exist,

(a)

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

2 Using polar coordinates we have $x = r\cos\theta$, $y = r\sin\theta$ so that

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} = \lim_{r^2\to 0} \frac{e^{-r^2}-1}{r^2} = \lim_{\alpha\to 0} \frac{e^{-\alpha}-1}{\alpha}$$

2

$$= \lim_{\alpha\to 0} \frac{(e^{-\alpha}-1)'}{(\alpha)'} = \lim_{\alpha\to 0} \frac{-e^{-\alpha}}{1} = -1.$$

1

2

(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

. Along the path $y=0$ we have

2
$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy^3}{x^2+y^6} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

. Along the path $x = y^3$ we have

2
$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \lim_{y\to 0} \frac{1}{2} =$$

1 These two limits are different so

2
$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$ does not exist.

(other paths are possible)

Exercise #7:(11 pts) The temperature of a body at a point (x, y, z) is given by

$$T = e^{xy} - xy^2 - x^2yz.$$

Determine the direction of the greatest drop in the temperature at the point $(1, -1, 2)$.

solution: The greatest drop (decrease) at

(4) the point $(1, -1, 2)$ is in the direction of the negative gradient at that point.

(4) $\nabla T = (ye^{xy} - y^2 - 2xyz)\,i + (xe^{xy} - 2xy - x^2z)\,j + (-x^2y)\,k$, so

$-\nabla T(1, -1, 2) = \langle e^{-1} - 3, -e^{-1}, -1 \rangle$

(3) is the direction of the greatest drop in the temperature at the point $(1, -1, 2)$.

Exercise #8 (14 pts) Identify the following surfaces and sketch each one in 3D-space,

(a) $x^2 + 4y + 9z^2 = 0$

$$-y = \frac{1}{4}x^2 + \frac{9}{4}z^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{z}{2/3}\right)^2$$

Elliptic paraboloid along the y-axis in the direction $y < 0$.

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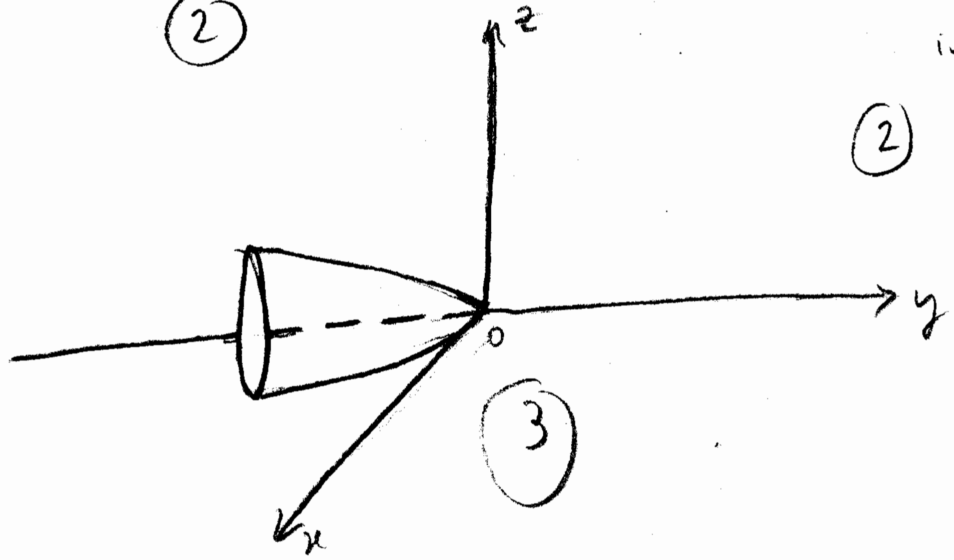

(b) $x^2 - 4y^2 + z^2 - 6x - 8y - 2z + 6 = 0$

$$x^2 - 6x - 4y^2 - 8y + z^2 - 2z + 6 = 0$$

$$[(x-3)^2 - 9] - 4(y+1)^2 + 4 + (z-1)^2 - 1 + 6 = 0$$

$$(x-3)^2 - \left(\frac{y+1}{\frac{1}{2}}\right)^2 + (z-1)^2 = 0$$

$$\left(\frac{y+1}{\frac{1}{2}}\right)^2 = (x-3)^2 + (z-1)^2$$

2

circular cone with vertex at (3, -1, 1

and axis along the y-axis

2