KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF MATHEMATICS AND STATISTICS

S121

MATH 201
MAJOR EXAM I
Wednesday October 03, 2012
Time: 8:30 PM-10:30 PM
Location: Bldg 54

NAME:.......Key...ID:................................SECTION..........

Instructions: Formula sheet, calculator and mobile are not allowed.

SHOW ALL YOUR WORK

Question #	1	2	3	4	5	6	7	8	9	Total
Points	12	10	10	15	15	8	8	12	10	100
Marks										

Exercise #1:(12pts) A curve C is defined by the parametric equations

$$x = 1 - t^3, y = 1 + t - t^2, t \in [0, 3]$$

(a) Find the point(s), if any, at which the normal line to the parametric curve C has slope 1.

(b) Determine where the curve is concave upward or downward.

(a) Since the slope of the normal line at a point is

2 $m_N = 1 \neq 0$, the slope of the tangent line to the curve at the same point is $m_T = -\frac{1}{m_N} = -1$.

We have $\frac{dx}{dt} = -3t^2$, $\frac{dy}{dt} = 1 - 2t$, so that,

$$m_T = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-2t}{-3t^2} = -1, \text{ giving,}$$

2 $3t^2 + 2t - 1 = 0$ whose solutions are

$t = -1 \notin [0, 3]$ and $t = \frac{1}{3} \in [0, 3]$. Therefore, there is only one point at which the normal line to the parametric curve C has slope 1

and the point is

2 $$(x, y) = \left(1 - \left(\tfrac{1}{3}\right)^3, 1 + \tfrac{1}{3} - \left(\tfrac{1}{3}\right)^2\right) = \left(\tfrac{26}{27}, \tfrac{11}{9}\right).$$

(b) We have

2 $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1-2t}{-3t^2}\right)}{-3t^2}$$ 2

$$= \frac{1}{(-3t^2)^3}\left[-2(-3t^2) - (-6t)(1-2t)\right] = \frac{6t^2 - 6t}{27t^6} = \frac{2}{9}\frac{t-1}{t^5}$$

so

2

t	0		1		3
$\frac{d^2y}{dx^2}$	‖	$-$	0	$+$	
Curve		concave down		concave up	

Exercise #2:(10pts) Find the length of the curve

$$x = \cos t + t\sin t, y = \sin t - t\cos t, t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

The length of the curve is given by

3

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

but $\frac{dx}{dt} = -\sin t + \sin t + t\cos t = t\cos t$

2

and $\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$

so

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(t\cos t)^2 + (t\sin t)^2}\, dt$$

2

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{t^2(\underbrace{\cos^2 t + \sin^2 t}_{1})}\, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|\, dt = 2\int_0^{\frac{\pi}{2}} t\, dt = t^2\Big|_0^{\frac{\pi}{2}}$$

2

1

$$= \frac{\pi^2}{4}.$$

Exercise #3:(10pts) Write the polar equation $r = 2\cos\theta + 2\sin\theta$ in cartesian coordinates, then describe and sketch the graph of the resulting equation (including the axes !).

Multiplication of both sides of the polar equation by r gives $r^2 = 2r\cos\theta + 2r\sin\theta$

2

but 2 $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$,

so

$$x^2 + y^2 = 2x + 2y$$

$$x^2 - 2x + y^2 - 2y = 0$$

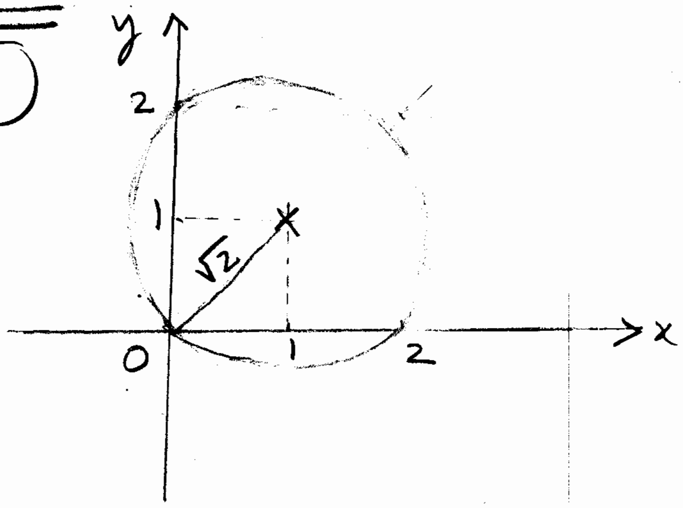
completing the squares yields

2 $$(x-1)^2 + (y-1)^2 = 2$$

which is an equation for the circle centered at $(1,1)$ with radius $\sqrt{2}$.

Note that the circle passes through the origin $(0,0)$, $(0,2)$, $(2,0)$

1



3

Exercise #4:(15pts)

(a) Sketch the curves $C_1 : r = 2\cos 2\theta$ and $C_2 : r = 1$ on the same axes. (b) Find the area inside the curve C_1 and outside the curve C_2 when $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

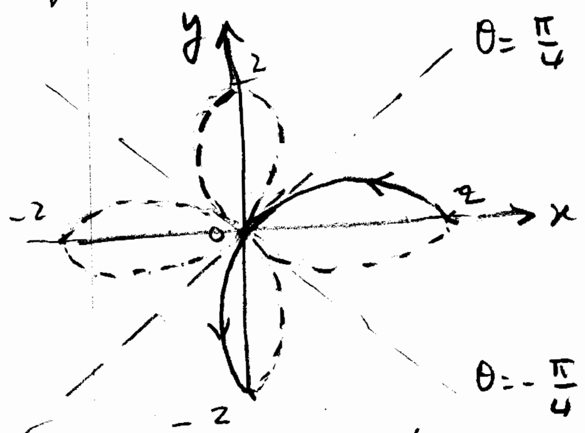
(a) $\mathcal{C}_1 : r = 2\cos 2\theta$

changing θ into $-\theta$ and θ into $\pi - \theta$ do

2 not change the value of r so the curve is symmetric with respect to the x-axis and the y-axis.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
2θ	$0 \nearrow$	$\frac{\pi}{2} \nearrow$	π
r	$2 \searrow$	$0 \searrow$	-2

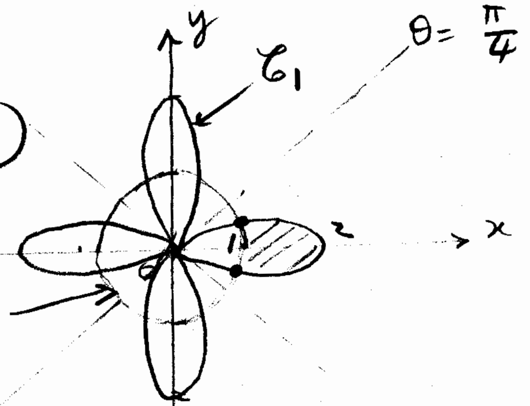
1



Using the symmetries, we get the four-leaved rose as seen above.

On the same axes,

5



(b) Intersection points:

2 $2\cos 2\theta = 1 \quad \therefore \cos 2\theta = \frac{1}{2}$

$\therefore 2\theta = \pm\frac{\pi}{3}$ that is

$\theta = \pm\frac{\pi}{6}$

So the area (of the hached region) is

2 $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[(2\cos 2\theta)^2 - (1)^2 \right] d\theta$

$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[4\cos^2 2\theta - 1 \right] d\theta = \int_0^{\frac{\pi}{6}} \left[4\,\frac{\cos 4\theta + 1}{2} - 1 \right] d\theta$

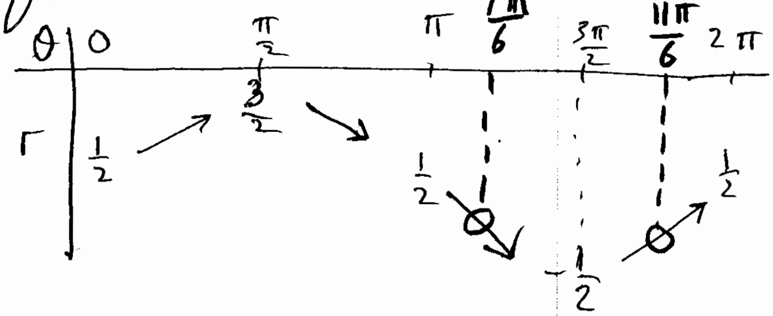
$= \int_0^{\frac{\pi}{6}} \left[2\cos 4\theta + 1 \right] d\theta = 2\,\frac{\sin 4\theta}{4} + \theta \Big|_0^{\frac{\pi}{6}} = \frac{1}{2}\sin\frac{2\pi}{3} + \frac{\pi}{6} \Big] = \frac{\sqrt{3}}{4} + \frac{\pi}{6}$

2 1

Exercise #5:(15pts)
(a) Graph $r = \frac{1}{2} + \sin\theta$, $\theta \in [0, 2\pi)$ showing all your steps.
(b) Find the slope of the tangent line to the graph at $\theta = 0$.

(a) The variations of r as a function of θ are as follows

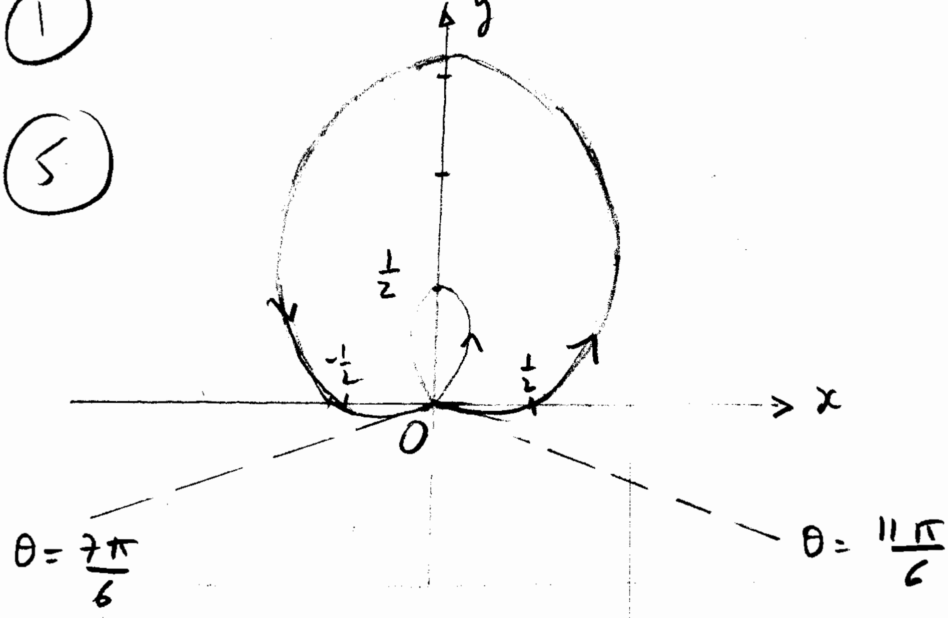
2



$\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$ have been obtained as values of θ for which $r = 0$ (r changes sign) 1 : $\frac{1}{2} + \sin\theta = 0$ $\therefore$ $\sin\theta = -\frac{1}{2}$ $\therefore$ $\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ 1

and $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$ 1

5



(b) The slope of the tangent line to the graph at $\theta = 0$ is

$$\left.\frac{dy}{dx}\right|_{\theta=0} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=0} = \left.\frac{\frac{d}{d\theta}\left[\left(\frac{1}{2}+\sin\theta\right)\sin\theta\right]}{\frac{d}{d\theta}\left[\left(\frac{1}{2}+\sin\theta\right)\cos\theta\right]}\right|_{\theta=0} = \left.\frac{\cos\theta\sin\theta + \left(\frac{1}{2}+\sin\theta\right)\cos\theta}{\cos^2\theta - \left(\frac{1}{2}+\sin\theta\right)\sin\theta}\right|_{\theta=0} = \frac{1}{2}.$$

2 2 1

Exercise #6:(8pts) Find an equation of the sphere which has one of its diameters having end points $(2, 1, 6)$ and $(4, 3, 10)$.

The center of the sphere is at

(2) $\left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{6+10}{2}\right) = (3, 2, 8)$.

The diameter of the sphere is

(2) $\sqrt{(4-2)^2 + (3-1)^2 + (10-6)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

(1) and it's radius is $\frac{2\sqrt{6}}{2} = \sqrt{6}$. So an equation for the sphere is

(3) $$(x-3)^2 + (y-2)^2 + (z-8)^2 = 6.$$

Remark: It is worth noting that any point $P(x, y, z)$ on the sphere with diameter AB where $A(2, 1, 6)$, $B(4, 3, 10)$ will satisfy $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ so that

$$(x-2)(x-4) + (y-1)(y-3) + (z-6)(z-10) = 0$$

constitutes an equation of the sphere, though it is not in standard form.

Exercise #7:(8pts) Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2) where $r_1, r_2 > 0$ and $0 \le \theta_1, \theta_2 < 2\pi$.

Let $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in polar coordinates. The cartesian coordinates would be

$P_1(r_1 \cos\theta_1, r_1 \sin\theta_1)$ 2 and $P_2(r_2 \cos\theta_2, r_2 \sin\theta_2)$ 2

giving the distance

$$d = \sqrt{(r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_2 \sin\theta_1)^2}.$$

4

Exercise #8:(12pts) Given $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + 6\vec{j} - 2\vec{k}$

(a) Find the angle between $\vec{a}$ and $\vec{b}$

(b) Find the vector projection of $\vec{a}$ onto $\vec{b}$.

(c) Find the scalar projection of $\vec{a}$ onto $\vec{b}$.

(a) $$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1)(3)+(2)(6)+(3)(-2)}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+6^2+(-2)^2}}$$

2

$$= \frac{3+12-6}{\sqrt{14}\sqrt{49}} = \frac{9}{7\sqrt{14}} \quad \therefore \theta = \cos^{-1}\left(\frac{9}{7\sqrt{14}}\right)$$

2

(b) $$\text{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$ 2

$$= \frac{(1)(3)+(2)(6)+(3)(-2)}{(3^2+6^2+(-2)^2)}(3\vec{i}+6\vec{j}-2\vec{k})$$

$$= \frac{9}{49}(3\vec{i}+6\vec{j}-2\vec{k}) = \frac{27}{49}\vec{i} + \frac{54}{49}\vec{j} - \frac{18}{49}\vec{k}$$ 2

(c) $$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(1)(3)+(2)(6)+(3)(-2)}{\sqrt{3^2+6^2+(-2)^2}} = \frac{9}{7}$$

2

2

Exercise #9:(10pts) Determine, using only the dot product, whether the triangle with vertices $P(-1,2,3), Q(2,-2,0)$ and $R(3,1,-4)$ is a right angle triangle.

1 1

We have $\overrightarrow{PQ} = \langle 3, -4, -3 \rangle$, $\overrightarrow{QR} = \langle 1, 3, -4 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, -7 \rangle$. Thus,

$$\overrightarrow{PQ} \cdot \overrightarrow{QR} = (3)(1) + (-4)(3) + (-3)(-4) = 3 - 12 + 12 = 3 \neq 0$$ 2

$$\overrightarrow{QR} \cdot \overrightarrow{PR} = (1)(4) + (3)(-1) + (-4)(-7) = 4 - 3 + 28 = 29 \neq 0$$ 2

$$\overrightarrow{PR} \cdot \overrightarrow{PQ} = (4)(3) + (-1)(-4) + (-7)(-3) = 12 + 4 + 21 = 37 \neq 0$$ 2

so none of the three angles of the triangle PQR is a right angle. Hence, PQR is not a right angle triangle.

2