KING FAHD UNIVERSITY OF PETROLEUM AND MINERAL

Department of Mathematical Sciences

Final Exam | MATH - 621 | Sem 112

Student #:______________ Name:_____________________

Show All Your Work. No Credits for Answers Not Supported by Work.

In this exam you can use any result in the book or notes. We use the symbols
$B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}, \quad S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}, \quad I = [0,1]$

Q1) (40 Points) Let X be the unit square in the plane ($X = I \times I$) with the dictionary order topology. That is $(x, y) < (u, v)$ iff $x < u$ or $x = u$ and $y < v$. Consider X with the order topology. Show that:

a. X is connected

b. X is not path connected

c. X is completely normal

d. X is Compact (Hint: use the fact that in the order topology, X is compact if it is complete)

e. X is first countable

f. X is not separable

g. X is Lindelöf

h. X is not metrizable

Q2) (10 Points) Consider the space $\mathbb{R}^{\omega}$. Is the space $\mathbb{R}^{\omega}$ normal in the product topology? In the uniform topology?

Q3) (10 Points) Let l^2 denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $x = (x_1, x_2, \cdots)$ such that $\sum x_i^2$ converges. (You may assume standard facts about infinite series). The l^2 metric is defined as follows, for any $\boldsymbol{x} = (x_i)$ and $\boldsymbol{y} = (y_i)$ we define

$$d(x, y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2 \right]^{1/2}.$$

Show that the l^2 space in the l^2 metric has a countable dense subset, and therefore has a countable base.

Q4) (10 Points) Let X be a locally compact Hausdorff space and let Y be the one-point compactification of X. Show that X is second countable iff Y is metrizable.

Q5) (10 Points) (a) Show that if A is a retract in B^2, then every continuous map $f : A \to A$ has a fixed point.

(b) Let X be a space homeomorphic to B^n show that any continuous map $f : X \to X$ has a fixed point.

Q6) (10 Points) Suppose that $h : S^1 \to S^1$ is nullhomotopic map.

a. Show that *h* has a fixed point.

b. Show that *h* maps some point *x* to its antipode $-x$.

Q7) (10 Points) Let $p : X \to Y$ be a covering map from a space *X* to a space *Y*.

a. Show that *p* is open map.
b. Let Y be connected. Show that if $p^{-1}(y_0)$ has *k* elements for some $y_0 \in Y$, then $p^{-1}(y)$ has *k* elements for each $y \in Y$, i.e., *X* is a *k*- fold cover of *Y*.

Q8) (30 Points) Let $A \subseteq X$ be retract of *X* and assume that $r : X \to A$ is a retraction. If $a_0 \in A$ show that

a. The induced map $r_*:\pi_1(X,a_0)\to\pi_1(A,a_0)$ is surjection (onto).

b. If $i:A\to X$ is the inclusion map, then the induced homomorphism $i_*:\pi_1(A,a_0)\to\pi_1(X,a_0)$ is injective.

c. Furthermore, if A is deformation retract of X, then i_* is an isomorphism.

d. Is S^1 a retract of B^2? Explain why.

e. Show that $S^0=\{-1,1\}$ is not retract of $B^1=[-1,1]$

f. Show that any continuous map $f:B^1\to S^0$ has a fixed point.

Q9) (10 Points) A space Y is said to the ***universal extension property*** (UEP) if for each triple consisting of a normal space X, a closed subset A of X, and a continuous function $f:A\to Y$, there exist an extension of f to a continuous map of X into Y.

(a) Show that $\mathbb{R}^J$ has a UEP.

(b) Show that if Y is homeomorphic to a retract of $\mathbb{R}^J$, then Y has the UEP.

Q10) (10 Points) Let Y be a normal space. Then Y is said to be an absolute retract if for every pair of spaces (Y_0, Z) such that Z is normal and Y_0 is closed subspace of Z homeomorphic to Y, the space Y_0 is retract of Z. Assume that Y is compact.

a. Show that if Y has the UEP then Y is an absolute retract .

b. Show that if Y is an absolute retract then Y has the UEP. [Assume the Tychonoff theorem, so you know $[0,1]^J$ is normal. Imbed Y in $[0,1]^J$.]

Total score 150