King Fahd Univ. of Petroleum and Minerals
Faculty of Sciences
Department of Mathematics and Statistics

Major 2
(MATH. 465-112)

Name:
ID:

Prob. 1

Consider the system $y' = A(t)y + g(t)$. Suppose that $A(t)$ and $g(t)$ are continuous for $-\infty < t < \infty$ and that $\int_{-\infty}^{+\infty} |A(t)|\, dt < \infty$ and $\int_{-\infty}^{+\infty} |g(t)|\, dt < \infty$. Show that the solution $\phi(t)$ exists for $-\infty < t < \infty$ and compute a bound for $|\phi(t)|$ valid for $-\infty < t < \infty$.

Prob. 2

Show that if a real homogeneous system of two first order equations has a fundamental matrix $\begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{pmatrix}$, then $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a also a fundamental matrix. Find another real fundamental matrix.

Prob. 3

Show that if all eigenvalues have real part negative or zero and if those eigenvalues with zero real parts are simple, there exists a constant $K > 0$ such that $\left|e^{tA}\right| \leq K$, $0 \leq t < \infty$ and hence every solution of $y' = Ay$ is bounded on $0 \leq t < \infty$.

Prob. 4

Sketch the phase portrait of $x'' - 2x' + x = 0$. Identify the origin and decide whether it is an attractor.

Prob. 5

Compute the Lipschitz constant K for $f(t, y) = t^2 + y^4$ in the region $\{(t, y) : |t| \leq 1, |y| \leq 3\}$.

Prob. 6

Construct the successive approximations to the solution ϕ of $y' = y$ that satisfies $\phi(0) = 1$.

Prob. 7

State and prove a uniqueness theorem for solutions of $y'' + g(t, y) = 0$, $y(0) = y_0$, $y'(0) = z_0$ where g is a given function defined on a rectangle

$$R = \{(t, y) : |t| \leq a, |y - y_0| \leq b\}.$$

Prob. 8

Consider the example: $y' = y^2$, $y(0) = 1$ studied in class. We found that solutions exist on $(-1/4, 1/4)$. Consider now the continuation of the solution ϕ to the right through the point $(1/4, 4/3)$. Show that on any rectangle

$$R = \{(t, y) : |t - 1/4| \leq a, |y - 4/3| \leq b\}$$

we have $M = \max_R y^2 = \left(\frac{4}{3} + b\right)^2$. Deduce that $\alpha_1 = 3/16$. This now gives existence on $-1/4 \leq t \leq 7/16$.)