King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 311
Final Exam – 2012–2013 (112)
Saturday, May 19, 2012

Allowed Time: 3 hours

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: ___________ Serial Number: ___________

Instructions:

1. Write **clearly** and **legibly**. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**

Question #	Grade	Maximum Points
1		09
2		17
3		12
4		09
5		11
6		22
Total:		80

Exercise 1:

Let φ be a continuous function defined on $\mathbb{R}$ and has the following properties:
(i) $\lim_{x \longrightarrow 0} \varphi(x) = 1$ and $\varphi(x_1 + x_2) = \varphi(x_1)\,\varphi(x_2), \ \forall\, x_1,\, x_2 \in \mathbb{R}$
Prove that:
a)- $\varphi(x) > 0, \forall\, x \in \mathbb{R}$.
b)- $\varphi(r\,x) = [\varphi(x)]^r$ for all rational number r.
c)- If $\varphi(1) = 1$ then φ is constant.

Exercise 2: A). Let $T_n(x) = \sum_{p=0}^{n} \frac{x^p}{p!}$.

Prove that: $T_n(x) < T_{n+1}(x) < e^x < \left(1 - \frac{x^{n+1}}{(n+1)!}\right)^{-1} T_n(x),\ \forall\, x > 0.$

B). i)-Use the Mean value theorem to show that: $1 - x < \cos x < 1 + x, \quad \forall\, x > 0.$

ii)- Find $\lim\limits_{x \longrightarrow 0} \dfrac{\cos x - 1}{x^2}$.

Exercise 3:

1. Let

$$f(x) = \begin{cases} -1 & if \quad x \quad is \quad irrational \\ 1 & if \quad x \quad is \quad rational \end{cases}$$

Show that f is not continuous anywhere.

2. Let $f(x) = 0$ if x is irrational and $f(\frac{p}{q}) = \frac{1}{q}$ if p and q are positive integers with no common factors. Show that f is discontinuous at every rational and continuous at every irrational on $(0, \infty)$.

Exercise 4: Let $f, g \in \mathcal{R}[a,b]$.

a)- If $t \in \mathbb{R}$, show that $\int_a^b (tf-g)^2(x)\,dx \geq 0$.

b)- Use a) to show that $2 \mid \int_a^b f(x)\,g(x)\,dx \mid \leq t \int_a^b f^2(x)\,dx + \frac{1}{t}\int_a^b g^2(x)\,dx, \quad t>0.$

c)- Prove the Cauchy-Shwarz inequality: $\left(\int_a^b \mid f(x)\,g(x) \mid \,dx\right)^2 \leq (\int_a^b f^2(x)\,dx)\,.(\int_a^b g^2(x)\,dx).$

Exercise 5:

a)- Let f be a continuous function and g, h two differentiable functions. Prove that:

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x)).h'(x) - f(g(x)).g'(x) \qquad \text{(a)}$$

b)- Use a) to find the derivatives of :

i. $f(x) = \displaystyle\int_{\sqrt{x}}^{x^3} \sin t\,dt$

ii. $g(x) = \displaystyle\int_{\tan x}^{x^2} \cos t\,dt$

Exercise 6:

Let $X = (x_n)$ be real sequence.

1. Show that if $\sum x_n$ is absolutely convergent then it is convergent.

2. Use the definition to prove that if $\lim\limits_{n \longrightarrow \infty} \dfrac{x_n - x}{x_n + x} = 0$ then $\lim\limits_{n \longrightarrow \infty} x_n = x \ , x \in \mathbb{R}$.

Hint: You may define $t_n = \frac{x_n - x}{x_n + x}$.

3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\ln(n+2) - \ln(n+1)}{\ln(n+1). \ln(n+2)}$$

converges or diverges. If it converges find its sum.

4. Show whether or not the series

$$\sum_{n=1}^{\infty} n\, e^{-2n}$$

converges or diverges. Justify your answer.