King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 302 Exam II

Semester (112) ~~April 15, 2012~~ Time: 6:15 - 7:45 pm

Name: Solution

I.D: .... Section: ....

Question	Points
1	____ / 10
2	____ / 14
3	____ / 12
4	____ / 14
Total	____ / 50

5 $\longrightarrow$ 15 mins

Question 1. Evaluate the line integral

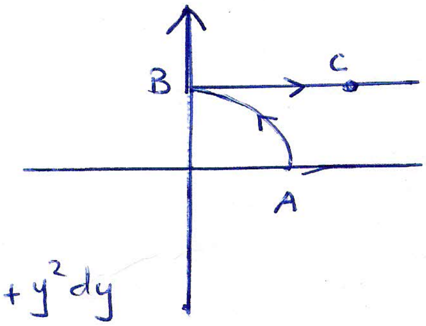
$$\int_C x^2y\,dx + y^2dy$$

where C is the curve consisting of the quarter circle $x^2+y^2=1$ in the XY-plane from $(1,0)$ to $(0,1)$, followed by the horizontal line from $(0,1)$ to $(2,1)$.

$C = C_1 \cup C_2$ such that

$C_1 = \{(\cos t, \sin t),\ 0 \le t \le \frac{\pi}{2}\}$

$C_2 = \{(t, 1)\ /\ 0 \le t \le 2\}$



$$\int_C x^2y\,dx + y^2dy = \int_{C_1} x^2y\,dx + y^2dy + \int_{C_2} x^2y\,dx + y^2dy$$

$$= \int_0^{\pi/2} \left(\cos^2 t \sin t(-\sin t)\,dt + \sin^2 t \cos t\,dt\right) + \int_0^2 t^2\,dt + 0$$

$$= -\int_0^{\frac{\pi}{2}} \left(\frac{\sin 2t}{2}\right)^2 dt + \int_0^{\pi/2} \sin^2 t \cos t\,dt + \left[\frac{t^3}{3}\right]_0^2$$

$$= -\frac{1}{4}\int_0^{\pi/4} \sin^2(2t)\,dt + \frac{1}{3}\left[\sin^3 t\right]_0^{\pi/2} + 8/3$$

$$= -\frac{1}{4}\int_0^{\pi/4} \frac{1-\cos 4t}{2}\,dt + \frac{1}{3} + 8/3$$

$$= -\frac{1}{8}\left[t - \overset{0}{\cancel{\frac{\sin 4t}{4}}}\right]_0^{\pi/4} + 3 = \boxed{3 - \frac{\pi}{16}}$$

Question 2. Given the vector field

$$F(x, y, z) = (z + ye^z)\,\mathbf{i} + (xe^z + 6y\sin(y^2))\,\mathbf{j} + x\,(1 + ye^z)\,\mathbf{k}.$$

(a) Show that F is conservative.

(b) Find a potential $\phi(x, y, z)$ of F.

(c) Compute $\int_C F.dr$, if $C = \{(\cos(t), \sin(t), t)\,,\ \ 0 \le t \le 2\pi\}$.

(a)

$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z+ye^z & xe^z+6y\sin(y^2) & x(1+ye^z) \end{vmatrix}$$

$$= (x e^z - xe^z)\vec{i} - ((1+ye^z) - (1+ye^z))\vec{j} + (e^z - e^z)\vec{K}$$

$= \vec{0}$. Therefore F is conservative since F and Curl F are continuous.

(b) A potential ϕ satisfies $\nabla\phi = F$. So,

$$\frac{\partial \phi}{\partial x} = z + ye^z \Rightarrow \phi = x(z+ye^z) + \psi(y,z)$$

$$\frac{\partial \phi}{\partial y} = \cancel{xe^z} + \frac{\partial \psi}{\partial y} = \cancel{xe^z} + 6y\sin(y^2)$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = 6y\sin(y^2) \Rightarrow \psi = -3\cos(y^2) + \alpha(z)$$

Thus $\phi = x(z+ye^z) - 3\cos(y^2) + \alpha(z)$

$$\Rightarrow \frac{\partial \phi}{\partial z} = x + \cancel{xye^z} + \alpha'(z) = x + \cancel{xye^z} \Rightarrow \alpha'(z) = 0$$

$\Rightarrow \alpha(z) =$ constant ($= 0$ for example)

A potential is $\boxed{\phi(x,y,z) = x(z+ye^z) - 3\cos(y^2)}$

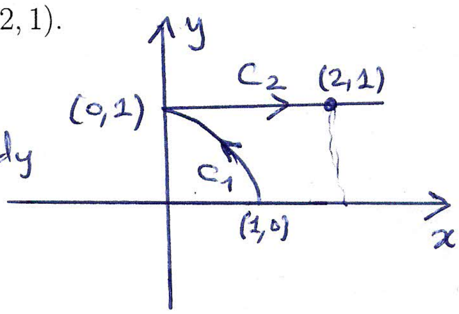
(c) $\int_C F.dr = \phi(B) - \phi(A) = \phi(1,0,2\pi) - \phi(1,0,0) =$

$= (2\pi - 3) - (0 - 3) = 2\pi$.

Question 1. Evaluate the line integral

$$\int_C x^2 y dx + y^2 dy$$

where C is the curve consisting of the quarter circle $x^2 + y^2 = 1$ in the XY-plane from $(1, 0)$ to $(0, 1)$, followed by the horizontal line from $(0, 1)$ to $(2, 1)$.



$$\int_C x^2 y\,dx + y^2 dy = \int_{C_1} x^2 y\,dx + y^2 dy + \int_{C_2} x^2 y\,dx + y^2 dy$$

(*) C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi/2$

$$\int_{C_1} x^2 y\,dx + y^2 dy = \int_0^{\pi/2} (-\sin^2 t \cos^2 t + \sin^2 t \cos t)\,dt$$

$$= -\int_0^{\pi/2} \left(\frac{\sin 2t}{2}\right)^2 dt + \int_0^{\pi/2} \sin^2 t \cos t\,dt$$

$\sin t = u$
$du = \cos t\,dt$

$$= -\frac{1}{4} \int_0^{\pi/2} \frac{1 - \cos 4t}{2}\,dt + \frac{1}{3} \sin^3 t \Big]_0^{\pi/2}$$

$$= -\frac{1}{8} \left[t - \cancel{\frac{\sin 4t}{4}} \right]_0^{\pi/2} + \frac{1}{3} = -\frac{\pi}{16} + \frac{1}{3}$$

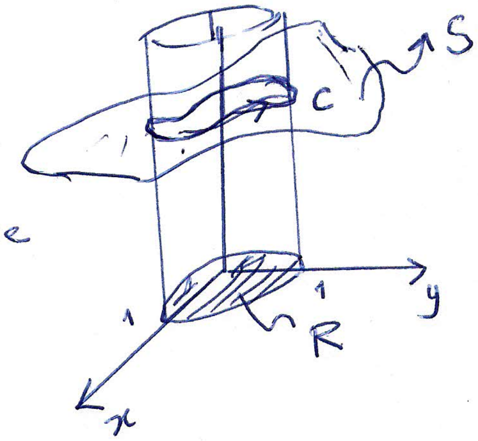
(**) $$\int_{C_2} x^2 y\,dx + y^2 dy = \int_0^2 x^2 (1)\,dx + 0 = \frac{x^3}{3} \Big|_0^2 = \frac{8}{3}$$

Therefore $$\int_C F \cdot dr = -\frac{\pi}{16} + \frac{1}{3} + \frac{8}{3} = 3 - \frac{\pi}{16}.$$

Question 4. Evaluate $\oint_C F.dr$ where

$$F(x,y,z) = x^2y\,\mathbf{i} + \frac{x^3}{3}\,\mathbf{j} + xy\,\mathbf{k}$$

and C is the curve of intersection of $z = y^2 - x^2$ and the cylinder $x^2+y^2 = 1$. Assume C is oriented counter-clockwise.



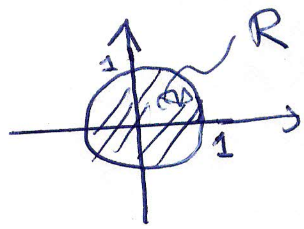
Rmk. The curve C is not a circle So it is hard to compute the line integral $\oint_C F.dr$. We then use

$$I = \oint_C F.dr = \iint_S \text{curl } F.\vec{n}\, dS \quad \text{(stokes' theorem)}$$

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & \frac{x^3}{3} & xy \end{vmatrix} = x\vec{i} - y\vec{j}.$$

The surface S is given by $\phi(x,y,z) = z + x^2 - y^2$

So $N = \nabla\phi = (2x, -2y, 1) \Rightarrow$ the unit normal is



$$n = \frac{1}{\sqrt{1+4(x^2+y^2)}}(2x, -2y, 1).$$

$z = f(x,y) = -x^2+y^2 \Rightarrow f_x = -2x,\ f_y = 2y.$

$$\text{So } I = \iint_R \frac{2(x^2+y^2)}{\cancel{\sqrt{1+4(x^2+y^2)}}} \cdot \cancel{\sqrt{1+f_x^2+f_y^2}}\, dA = \iint_R 2(x^2+y^2)\, dA$$

Using polar coordinates, we get

$$I = 2\int_0^{2\pi}\int_0^1 r^2\, r\, dr\, d\theta = 2\int_0^{2\pi} d\theta \int_0^1 r^3 dr = 4\pi\,\frac{1}{4} = \pi.$$