KFUPM – Department of Mathematics and Statistics – Term 112

MATH 280

Final Exam (May 23, 2012)

Duration: 3 Hours

NAME:_________________________ ID:_________________________ Section: _01_

Exercise 1 (20 points)

Use the augmented matrix to find all values of r for which the system (S) has:

a/ No solution b/ a unique solution c/ infinitely many solutions

$$(S)\ \left\{\begin{array}{l} 2x + y + z = 2 \\ x - y + r^2 z = r \\ x + y - z = 2 \end{array}\right\}$$

Exercise 2 (20 points)

Let $L : R^4 \to R^4, L\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c \\ b-d \\ a \\ d-b \end{pmatrix}$

2-Find $KerL$ and $\dim KerL$. Is L a one-to-one linear operator?

3-Find $RangeL$ and $\dim(RangeL)$. Is L an onto linear operator?

Exercise 3 (20 points)

Let P_n be the vector space of all real polynomials of degree $\leq n$ and $D: P_4 \to P_3, D(f) = f'$ be the differential operator. Let $S = \{1, t, t^2, t^3, t^4\}$ and $T = \{1, t, t^2, t^3\}$ be the standard bases of P_4 and P_3 respectively. Set $S' = \{2, 1-t, -t^2, t^2 - t^3, t^4 - t^2\}$ and $T' = \{1, 1-t, 1-t^2, t^3\}$.

1-Prove that S' and T' are bases of P_4 and P_3 respectively.

2-Find the transition matrix P from S' to S.

3-Find the transition matrix Q from T' to T.

4-Find the matrix representing D with respect to S' and T'

Exercise 4 (20 points)

Use Gram-Schmidt process to transform the basis $S = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \}$ to an orthonormal basis.

Exercise 5 (20 points)

Let $A=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$ and set $W=\{B\in M_3(R)\mid AB=BA\}$.

1-Prove that W is a vector space.

2-Find a basis for W and $\dim W$.

3-Find the value of the positive integer n such that W is isomorphic to R^n.

Exercise 6 (20 points)

Let A and B be two $n \times n$ similar matrices (i. e. $B = P^{-1}AP$).

1-Prove that if λ_1 and λ_2 are distinct eigenvalues of A and V_1 and V_2 are eigenvectors associated to λ_1 and λ_2 respectively, then V_1 and V_2 are linearly independent.

2-Prove that A and B have the same eigenvalues.

3-Prove that for a common eigenvalue λ of A and B, if V is an eigenvector of A associated to λ, then $P^{-1}V$ is an eigenvector of B associated to λ.

Exercise 7 (20 points)

Let A be an $n\times n$ matrix with characteristic polynomial $f = X^n + a_{n-1}X^{n-1} + ... + a_1 X + a_0$ and suppose that $a_0 \neq 0$.

1-Prove that A is invertible and find A^{-1}.

2-Application: Find the 3×3 matrix A such that $f = X^3 - 4X^2 + 4X - 1$ and

$$A^2 - 4A = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -3 & 0 \\ -1 & 0 & -3 \end{pmatrix}$$

Exercise 8 (20 points)

Let L be a linear operator of R^3 whose matrix in the standard basis S is

$$[L]_S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

1-Prove that L is diagonalizable.

2-Find an orthonormal basis T such that $[L]_T = D$ is a diagonal matrix.

Exercise 9 (20 points)
Let A be a real symmetric matrix.
1-Prove that the eigenvalues of A are all real numbers.
2-Prove that eigenvectors associated to distinct eigenvalues are orthogonal.

Exercise 10 (20 points)

Let $g(x,y,z)=x^2+y^2+z^2+4xy+4xz-4yz$ be a quadratic form of R_3.

1-Find the canonical quadratic form h that is equivalent to g.

2-Find the rank and the signature of g