Department of Mathematics and Statistics, KFUPM
Math 280, Term 112
Exam 2, Due by April 4, 2012
Name and ID

Exercise 1(10 points).
Let $V = \mathbb{R}^3$ and $S = \{U_1, U_2, U_3\}$ its standard basis.
(1) Find a basis $T = \{V_1, V_2, V_3\}$, $T \neq S$ and a 3×3 matrix A such that $\{AV_1, AV_2, AV_3\}$ is not a basis for V.
(2) Find a 3×3 matrix A such that $\{AV_1, AV_2, AV_3\}$ is a basis for V.

Exercise 2(10 points).
Let V be the vector space of all continuous real-valued functions and W the subspace of V spaned by $\{cost, sint, t\}$.
(1) Find a basis for W.
(2) Find $dimW$.

Exercise 3(10 points).
Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ be the vector space of all $n \times n$ matrices and A a fixed $n \times n$ matrix. Let $\phi_A : V \to \mathbb{R}$ defined by $\phi_A(B) = Tr(AB)$ where $Tr(M)$ is the trace of the matrix M.
(1) Prove that ϕ_A is a homomorphism.
(2) Is ϕ_A an isomorphism?

Exercise 4 (10 points).
Let V and W be two vector spaces over a field K and let $\phi : V \to W$ be an isomorphism.
(1) Prove that $\phi^{-1} : W \to V$ is an isomorphism.
(2) Prove that if $S = \{U_1, \ldots, U_n\}$ is a basis for V, then $\phi(S) = \{\phi(U_1), \ldots, \phi(U_n)\}$ is a basis for W.
(3) Prove that $dimV = dimW$.

Exercise 5 (10 points).
Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ be the vector space of all $n \times n$ matrices and W the subset of V consisting of all symmetric matrices.
(1) Prove that W is a vector space.
(2) Find $dimW$.
(3) As an application, Find a basis for W when $n = 3$.

Exercise 6(10 points).
Let $V = \mathbb{P}_2 = \{f \in K[X] | deg(f) \leq 2\}$ and set $S = \{1, X, X^2\}$ and $T = \{1 - X, 1 + X, 1 - X^2\}$.
(1) Prove that S and T are bases for V.
(2) Find the transition matrix from S to T.

Exercise 7(10 points).
Let $V = \mathbb{R}_3$, $S = \{U_1 = (0,1,1), U_2 = (1,0,1), U_3 = (1,1,0)\}$ be a basis for V and set $V_1 = (1,-1,1), V_2 = (0,1,2)$ and $V_3 = (0,0,2)$.
(1) Verify that $\{V_1, V_2, V_3\}$ are linearly independent.
(2) Verify that the coordinate vectors $[V_1]_S, [V_2]_S$, and $[V_3]_S$ are linearly independent.
Generalizations. Prove that if W is an n-dimensional vector space, T is a basis for W and $W_1, \ldots, W_n$ are linearly independent, then the coordinate vectors $[W_1]_T, \ldots, [W_n]_T$ are linearly independent.

Exercise 8(10 points).

Let A be an $n \times n$ non-singular matrix and B be an $n \times m$ matrix with $rank(B) = p$.

(1) Find $rank(AB)$.

(2) Set $M = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 5 \\ 3 & 2 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 3 & 0 \end{pmatrix}$

Find $rank(N)$ and $rank(MN)$.

Exercise 9(10 points).
Let V be the vector space of all real-valued continuous functions, V_{even} its subspace of all even functions and V_{odd} its subspace of all odd functions.
(1) Prove that every element $f \in V$ can be written as $f = f_1 + f_2$ where $f_1 \in V_{even}$ and $f_2 \in V_{odd}$.
(2) Prove that $V_{even} \cap V_{odd} = \{0\}$.

Exercise 10(10 points).
Let V be the vector space of all real-valued continuous functions on the closed interval $[a, b]$ and consider the map $\phi : (V \times V) \longrightarrow \mathbb{R}$ defined by $\phi(f, g) = \int_a^b f(t)g(t)dt$.
(1) Prove that ϕ is an inner product.
(2) Find the distance from $f(t) = t$ to $g(t) = e^t$.