Version 1

King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 260

Final Exam, Semester II, 2011-2012

Saturday May 26, 2012

Net Time Allowed: 180 minutes (7:30am-10:30am)

Name:—— Key Solution

ID:—— Section:——

M.RF

Q#	Marks	Maximum Marks
1	15	15
2	15	15
3	25	25
4	15	15
5	10	10
6	15	15
7-15	45	45
Total	140	140

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Calculators and mobile phones are NOT allowed in this exam.
6. Turn off your mobile.

Note:
For Part II you should write your answers in the box below.

Part II

7	8	9	10	11	12	13	14	15
D	C	B	A	A	D	E	D	B

Part I

1. Find the general solution of the differential equation 15 pts

$$y'' + 9y = 2\cos 3x + 3\sin 3x.$$

$\boxed{1}$ $y'' + 9y = 0$

$\lambda^2 + 9 = 0$ so $\lambda = \pm 3i$ 2 pts

so $y_H = c_1 \cos 3x + c_2 \sin 3x$ 2 pts

Thus $y_p = x(A\cos 3x + B\sin 3x)$

$y_p' = x(-3A\sin 3x + 3B\cos 3x) + A\cos 3x + B\sin 3x$

$y_p'' = x(-9A\cos 3x - 9B\sin 3x) - 6A\sin 3x + 6B\cos 3x$

8 pts

$y_p'' + 9y_p = -9xA\cos 3x - 9xB\sin 3x - 6A\sin 3x + 6B\cos 3x$
$+ 9xA\cos 3x + 9xB\sin 3x$

$= -6A\sin 3x + 6B\cos 3x$

$= 2\cos 3x + 3\sin 3x$ L.H.S

so $-6A = 3$ i.e $A = -\frac{1}{2}$

$6B = 2$ i.e $B = \frac{1}{3}$

so $y_p = x(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x)$

so $y = c_1 \cos 3x + c_2 \sin 3x + x(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x)$ 3 pts

15 pts

2. Use the method of variation of parameters to find a particular solution of

$$(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2.$$

Hint: ($y_1 = x$ and $y_2 = 1 + x^2$ form a fundamental set of solutions for the associated homogeneous differential equation.)

Standard form:

$$y'' - \frac{2x}{x^2-1} y' + \frac{2}{x^2-1} y = x^2 - 1 \quad 2 \text{ pts}$$

$$W = \begin{vmatrix} x & 1+x^2 \\ 1 & 2x \end{vmatrix} = x^2 - 1 \quad 2 \text{ pts}$$

$$W_1 = \begin{vmatrix} 0 & 1+x^2 \\ x^2-1 & 2x \end{vmatrix} = -(1+x^2)(x^2-1) = (1+x^2)(1-x^2) \quad 2 \text{ pts}$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & x^2-1 \end{vmatrix} = x(x^2-1) \quad 2 \text{ pts}$$

$$u_1' = \frac{W_1}{W} = -1 - x^2 \Rightarrow u_1 = -x - \frac{x^3}{3} \quad 2 \text{ pts}$$

$$u_2' = \frac{W_2}{W} = x \Rightarrow u_2 = \frac{x^2}{2} \quad 2 \text{ pts}$$

$$y_p = u_1 y_1 + u_2 y_2 = x\left(-x - \frac{x^3}{3}\right) + (1+x^2)\left(\frac{x^2}{2}\right)$$

$$y_p = \frac{-x^2}{2} + \frac{x^4}{6} \quad 3 \text{ pts}$$

3. Let $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$.

4 pts (a) Show that the characteristic polynomial of A is: $P_A(\lambda) = (\lambda - 1)(\lambda - 2)^2$

14 (b) Find the eigenvalues and bases for the corresponding eigenspaces of A.

(c) Is A diagonalizable? 3 pts

(d) Find the Jordan canonical form of A. 4 pts

(a) $P_A(\lambda) = \begin{vmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 5 & \lambda+2 \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4$ ← 2 pts

$= (\lambda-1)(\lambda-2)^2$ ← 1

1

(b) The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$

• Eigenspace associated with $\lambda = 1$

$A - I_3 = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix}$

$A - I_3 \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & -3 & -1 \end{bmatrix}$

$\xrightarrow{R_{2,3}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so

$E_1 = \ker(A - I_3)$ is the set of all $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s.t. $x = -\frac{4}{3}z$, $y = -\frac{1}{3}z$; taking $z = -3$

$u_1 = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$ is the basis of E_1

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7 pts

• Eigenspace associated with $\lambda = 2$

$(A - 2I_3)v = 0$, ... $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$?

$(A - 2I_3) = \begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix}$

$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix}$

$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = -\frac{3}{2}z$, $y = -\frac{1}{2}z$

Taking $z = -2$, we obtain

$u_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ as a basis of E_2.

7 pts

(c) $\dim(E_1) + \dim(E_2) = 1 + 1 = 2 \neq 3$

$\Rightarrow$ A is not diagonalizable 3 pts

(d) As A is not diagonalizable, the Jordan canonical form of A is

$JCF(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4 pts

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4. Solve the following system: $Y' = AY$, where

15 pts

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$y_1' = y_1 + y_2$ (1)

$y_2' = y_2 - y_3$ (1)

$y_3' = 2y_3$ (1)

So $y_3 = c_1 e^{2x}$ (2) then $y_2' = y_2 - c_1 e^{2x}$

i.e. $y_2 = c_2 e^{x} - c_1 e^{2x}$ (3)

So $y_1' = y_1 + c_2 e^{x} - c_1 e^{2x}$

i.e. $y_1 = e^{x}(c_2 x - c_1 e^{x}) + c_3 e^{x}$

$= c_2 x e^{x} - c_1 e^{2x} + c_3 e^{x}$ (3)

Thus $M = \begin{bmatrix} -e^{2x} & xe^{x} & e^{x} \\ -e^{2x} & e^{x} & 0 \\ e^{2x} & 0 & 0 \end{bmatrix}$ (2)

$\Rightarrow$ The general sol^{n} is

$y = \begin{bmatrix} -e^{2x} & xe^{x} & e^{x} \\ -e^{2x} & e^{x} & 0 \\ e^{2x} & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ (2)

5. Convert the following system of differential equations into a system of first order differential equations in matrix form.

$$x'' = 5x - 4y$$
$$y'' = -4x + 5y + \sin t$$

3 pts

$$\left(\begin{array}{l|l} v_1 = x & v_1' = v_2 \\ v_2 = x' & v_2' = 5v_1 - 4v_3 \\ v_3 = y & v_3' = v_4 \\ v_4 = y' & v_4' = -4v_1 + 5v_3 + \sin t \end{array}\right) \leftarrow 4 \text{ pts.}$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin t \end{bmatrix}}_{3 \text{ pts}}$$

6. Let α, β be real numbers. Consider the initial value problem

$$y'' + \alpha y' + \beta y = 0, \quad y(0) = 3,\ y'(0) = 5.$$

Suppose that the differential equation has a fundamental set of solutions $\{y_1, y_2\}$, with $y_1 = e^{-x}$ and the Wronskian $\omega(y_1, y_2) = 4e^{2x}$.

(a) Show that e^{3x} is a solution to the differential equation. 5 pts

(b) Determine α, β. 5 pts

(c) Solve the initial value problem. 5 pts

(a) $W(y_1, y_2) = \begin{vmatrix} e^{-x} & y_2 \\ -e^{-x} & y_2' \end{vmatrix} = 4e^{2x} \Rightarrow e^{-x} y_2' + e^{-x} y_2 = 4e^{2x}$

$\Rightarrow y_2' + y_2 = 4e^{3x}$ 2 pts

I.F $\mu(x) = e^{\int dx} = e^{x}$

$\Rightarrow y_2 = \frac{1}{e^x}\left(\int e^x 4e^{3x}\,dx + C\right) = \frac{1}{e^x}\left(\int 4e^{4x}dx + C\right)$

$= e^{3x} + Ce^{-x} = e^{3x} + C y_1$ 2 pts

As y_1 & y_2 are solutions to the given homogenous DE and the set of solutions is a vector space, we deduce that $e^{3x} = y_2 - Cy_1$ is a solution. 1

(b) e^{-x}, e^{3x} are solutions $\Rightarrow \begin{cases} (-1)^2 - \alpha + \beta = 0 \\ 9 + 3\alpha + \beta = 0 \end{cases} \Rightarrow \begin{matrix} -\alpha + \beta = -1 \\ 3\alpha + \beta = -9 \end{matrix}$

$\Rightarrow \boxed{\alpha = -2}$ & $\boxed{\beta = -3}$ 5 pts

(c) The general solution to the given DE is

1 $y = c_1 e^{-x} + c_2 e^{3x}$, where c_1, c_2 are constants

initial conditions $\Rightarrow \left.\begin{matrix} c_1 + c_2 = 3 \\ -c_1 + 3c_2 = 5 \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} c_2 = 2 \\ c_1 = 1 \end{matrix}}$ 2

It follows that the unique solution to the given IVP is $\boxed{y = e^{-x} + 2e^{3x}}$. 2

7. A tank initially contains 600 liters of salt water with concentration $\frac{1}{15}$ kg/L. Suppose that a solution of salt water with concentration $\frac{1}{5}$ kg/L flows into the tank at a rate of 25 L/min. The solution in the tank is kept uniform. The solution flows out of the tank at a rate of 50 L/min. The amount of a salt in the tank at the end of 18 minutes is equal to:

   (a) 20 Kg
   (b) 41 Kg
   (c) 35 Kg
   (d) 25 Kg
   (e) 36 Kg

8. Which statements among the following is TRUE?

   (a) $\{(x, y) \in \mathbb{R}^2 | \ xy = 1\}$ is a subspace of $\mathbb{R}^2$.
   (b) $\{(x, y) \in \mathbb{R}^2 | \ 2x - y = 1\}$ is a subspace of $\mathbb{R}^2$.
   (c) $\{(x, y) \in \mathbb{R}^2 | \ x - 2y = 0\}$ is a subspace of $\mathbb{R}^2$.
   (d) $\{(x, y) \in \mathbb{R}^2 | \ xy = 0\}$ is a subspace of $\mathbb{R}^2$.
   (e) $\{(x, y) \in \mathbb{R}^2 | \ x^2 = y^2\}$ is a subspace of $\mathbb{R}^2$.

9. Let A be a 3×3 matrix such that $\text{rank}(A) = 1$. Then:

(a) $\dim(RS(A)) = 2$.

(b) $\dim(NS(A)) = 2$.

(c) A is a nonsingular matrix.

(d) $\dim(CS(A)) = 2$.

(e) $\text{rank}(2A) = 2$.

10. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}$.

(a) $\det(A) = 12$.

(b) $\det(A) = 9$.

(c) $\det(A) = 8$.

(d) $\det(A) = 6$.

(e) $\det(A) = 5$.

11. Let A be a 3×3 matrix such that $A^2 - 2A + 13I_3 = 0$. Then:

(a) A is a nonsingular matrix.

(b) $\text{rank}(A) < 3$.

(c) The reduced echelon form of A is $A_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) There exists $\lambda \in \mathbb{R}$ such that $A = \lambda I_3$.

(e) $\det(A - 2I_3) = 0$.

12. Let $S = \{v_1, v_2, v_3\}$ where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

Then:

(a) S is linearly independent but does not span $\mathbb{R}^3$.

(b) S spans $\mathbb{R}^3$ and S is linearly dependent.

(c) S does not span $\mathbb{R}^3$ and S is linearly dependent.

(d) S forms a basis for $\mathbb{R}^3$.

(e) None of the above.

13. The solution of the linear first order differential equation

$$(x^2+1)\frac{dy}{dx}+3xy=6x, \quad y(0)=3.$$

at $x=2$ is:

(a) $2+(5)^{-3/2}$
(b) $2+2(5)^{3/2}$
(c) 0
(d) $2+3(5)^{3/2}$
(e) $2-(5)^{-3/2}$

14. The value of the general solution at $x=1$ of the Bernoulli differential equation

$$\frac{dy}{dx}+(\frac{6}{x})y=3y^{4/3}$$

is:

(a) $y(1)=\frac{1}{C^3}$
(b) $y(1)=C^3$
(c) $y(1)=\frac{1}{(C-1)^3}$
(d) $y(1)=\frac{1}{(1+C)^3}$
(e) 0

15. Given that $y_1 = x\cos(\ln x)$ is a solution of the differential equation

$$x^2y'' - xy' + 2y = 0.$$

The value of the second solution y_2 (Using reduction of order) at $x = e$ is qual to:

(a) 0

(b) $e\sin 1$

(c) πe

(d) 1

(e) $e\cos 1$