Sol 1

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Math 260-01
Quiz 1

Name: ....

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Exercise 1. Determine the values $a, b \in \mathbb{R}$ so that the system

$$(\star)\quad \begin{cases} -2x + y + 3z = 4 \\ -3x + 2y + az = 5 \\ -4x + 3y + 5z = b \end{cases}$$

is consistent. In each case solve the system.

Solution:

$$\left(\begin{array}{ccc|c} -2 & 1 & 3 & 4 \\ -3 & 2 & a & 5 \\ -4 & 3 & 5 & b \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3-a & -1 \\ -3 & 2 & a & 5 \\ -4 & 3 & 5 & b \end{array}\right)$$

$$\xrightarrow[R_3+4R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3-a & -1 \\ 0 & -1 & 9-2a & 2 \\ 0 & -1 & 17-4a & b-4 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3-a & -1 \\ 0 & -1 & 9-2a & 2 \\ 0 & 0 & 8-2a & b-6 \end{array}\right)$$

Two cases have to be considered

Case 1: Suppose that $8-2a=0$ (that is $a=4$). In this case the system is consistent iff $b=6$

Here, the last augmented matrix is $\left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the system is equivalent to $\begin{cases} x - 2z = -3 \\ y - z = -2 \end{cases}$

This gives $x = -3 + 2z$ and $y = -2 + z$, with $z \in \mathbb{R}$.

Case 2. Suppose that $a \neq 4$. Then using back substitution,

we get $z = \frac{b-6}{8-2a}$ (sol 2)

$$y = -2 + (9-2a)z = -2 + (9-2a)\left(\frac{b-6}{8-2a}\right)$$

$$x = y - (3-a)z - 1 = -2 + \frac{(9-2a)(b-6)}{8-2a} - \frac{(3-a)(b-6)}{8-2a} - 1$$

$$= -3 + \frac{(6-a)(b-6)}{8-2a}.$$

Sol 3

Exercise 2. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & 1 & 1 \end{pmatrix}$.

(1) Find the inverse of A.

(2) Use the inverse of A to solve the following system of linear equations:

$$\begin{cases} x + y + z = 1 \\ 2x + z = 2 \\ 4x + y + z = 3 \end{cases}$$

Solution ①

$$[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -3 & -3 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & -1 \\ 0 & -3 & -3 & -4 & 0 & 1 \end{array}\right) \xrightarrow[R_3 + 3R_2]{R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 & 3 & -2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 2 & 1 & -1 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & -\frac{2}{3} \end{array}\right) \xrightarrow[R_2 - 2R_3]{R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 1 & -\frac{2}{3} \end{array}\right)$$

Hence, A is invertible and $A^{-1} = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \\ \frac{2}{3} & 1 & -\frac{2}{3} \end{pmatrix}$

② The solution to the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} + 3 \times \frac{1}{3} \\ \frac{2}{3} - 2 + 3 \times \frac{1}{3} \\ \frac{2}{3} + 2 - 3 \times \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Sol 4

Exercise 3. Express the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & 1 & 1 \end{pmatrix}$.
as a product of elementary matrices.

Solution: According to Exercise 2, we have the following sequence of elementary row operations

$$A \xrightarrow{R_2 - 2R_1} A_1 \xrightarrow{R_3 - 4R_1} A_2 \xrightarrow{R_2 - R_1} A_3 \xrightarrow{R_1 - R_2} A_4 \xrightarrow{R_3 + 3R_2} A_5 \xrightarrow{\frac{1}{3}R_2} A_6 \xrightarrow{R_1 + R_3} A_7$$

$$\xrightarrow{R_2 - 2R_3} A_8 = I_3.$$

To each elementary row operation, we assign an elementary matrix.

$A_1 = E_1 A$, with $I_3 \xrightarrow{R_2 - 2R_1} E_1$

$A_2 = E_2 A_1$, with $I_3 \xrightarrow{R_3 - 4R_1} E_2$

$A_3 = E_3 A_2$, with $I_3 \xrightarrow{R_2 - R_1} E_3$

$A_4 = E_4 A_3$, with $I_3 \xrightarrow{R_1 - R_2} E_4$

$A_5 = E_5 A_4$, with $I_3 \xrightarrow{R_3 + 3R_2} E_5$

$A_6 = E_6 A_5$, with $I_3 \xrightarrow{\frac{1}{3}R_2} E_6$

$A_7 = E_7 A_6$, with $I_3 \xrightarrow{R_1 + R_3} E_7$

$A_8 = E_8 A_7$, with $I_3 \xrightarrow{R_2 - 2R_3} E_8$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1}$$

It suffices to precise each elementary matrix E_i^{-1}

Sol 5

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_2 + 2R_1} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \xleftarrow{R_3 + 4R_1} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_2 + R_1} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_1 + R_2} I_3$$

$$E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xleftarrow{R_3 - 3R_2} I_3$$

$$E_6^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{3R_2} I_3$$

$$E_7^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_1 - R_3} I_3$$

$$E_8^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_2 + 2R_3} I_3$$