MATH 202.18 (Term 112)

Quiz 5 (Sect. 6.1) Duration: 20mn

Name: **ID number:**

Let $(x^2+1)y''+xy'+y=0$ **be a DE.**

a.)(3pts)Show that $x=0$ **is an ordinary point for the DE.**

b.)(7pts)Find two power series solutions in the form $y=\sum_{n=0}^{\infty}c_nx^n$**.**

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a) $y''+\frac{x}{x^2+1}y'+\frac{1}{x^2+1}y=0$

$P(x)=\frac{x}{x^2+1}$, $Q(x)=\frac{1}{x^2+1}$.

Both functions $P(x)$ and $Q(x)$ are analytic at $x=0$. Therefore, $x=0$ is an ordinary point of the DE

b) there is a solution in the form $y=\sum_{n=0}^{\infty}c_nx^n$, $|x|<1$.

We substitute y into the DE.

$$(x^2+1)\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+x\sum_{n=1}^{\infty}c_nnx^{n-1}+\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)c_nx^n+\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}+\sum_{n=1}^{\infty}c_nx^n+\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{k=2}^{\infty}c_kk(k-1)x^k+\sum_{k=0}^{\infty}c_{k+2}(k+1)(k+2)x^k+\sum_{k=1}^{\infty}kc_kx^k+\sum_{k=0}^{\infty}c_kx^k=0$$

$$2c_2+c_0+(6c_3+2c_1)x+\sum_{k=2}^{\infty}\left[c_k(k^2+1)+c_{k+2}(k+1)(k+2)\right]x^k=0$$

$$\begin{cases}2c_2+c_0=0\\6c_3+2c_1=0\\c_k(k^2+1)+c_{k+2}(k+1)(k+2)=0,\ k=2,\dots\end{cases}$$

$$\begin{cases}c_2=-\frac{c_0}{2}\\c_3=-\frac{c_1}{3}\\c_{k+2}=-\frac{(k^2+1)\,c_k}{(k+1)(k+2)},\ k=2,3,\dots\end{cases}$$

$$c_4=-\frac{5}{3.4}c_2=+\frac{5c_0}{2.3.4}$$

$$c_5=-\frac{10}{4.5}c_3=\frac{c_1}{3.2}$$

$$y=c_0+c_1x+c_2x^2+c_3x^3+\cdots$$

$$=c_0+c_1x-\frac{c_0}{2}x^2-\frac{c_1}{3}x^3+\frac{c_0}{2.3}x^4+\frac{2c_1}{3.5}x^5+\cdots$$

$$=c_0\underbrace{\left(1-\frac{x^2}{2}+\frac{5x^4}{24}+\cdots\right)}_{y_1}+c_1\underbrace{\left(x-\frac{x^3}{3}+\frac{1}{6}x^5+\cdots\right)}_{y_2}$$

$$y=c_1y_1+c_2y_2$$

MATH 202.15 (Term 112)

Quiz 5 (Sect. 6.1) Duration: 20mn

Name: ID number:

Let $(x+1)y'' - xy = 0$ be a DE.

a.)(2pts)Show that $x = 0$ is an ordinary point for the DE.

b.)(5pts)Find a relation of recurrence on c_n such that the series $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution to the DE.

c.)(3pts)Suppose that $c_0 = 0$ and $c_1 = 1$. Find the values of c_2, c_3 and c_4.

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a) $y'' - \frac{x}{x+1} y = 0$

$P(x) = 0$, $Q(x) = -\frac{x}{x+1}$

$P(x)$ and $Q(x)$ are both analytic at $x=0 \Rightarrow x=0$ is an ordinary point for the DE

b) There exist 2 solutions in the form $y = \sum_{n=0}^{\infty} c_n x^n$, $|x|<1$

We substitute y into the DE.

$$(x+1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-1} + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\sum_{k=1}^{\infty} c_{k+1} k(k+1)x^k + \sum_{k=0}^{\infty} c_{k+2}(k+1)(k+2)x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty}\left[k(k+1)c_{k+1} + c_{k+2}(k+1)(k+2) - c_{k-1}\right]x^k = 0$$

$$\begin{cases} 2c_2 = 0 \\ k(k+1)c_{k+1} + (k+1)(k+2)c_{k+2} - c_{k-1} = 0, \quad k=1,\dots \end{cases}$$

$$\begin{cases} c_2 = 0 \\ c_{k+2} = \frac{c_{k-1}}{(k+1)(k+2)} - \frac{k}{k+2} c_{k+1}, \quad k=1,2,\dots \end{cases}$$

c) $c_0 = 0$, $c_1 = 1$, $\boxed{c_2 = 0}$

$c_3 = \frac{c_0}{6} - \frac{c_2}{3} = 0$

$c_4 = \frac{c_1}{3\cdot 4} - \frac{1}{2}c_3 = \frac{1}{12}$