MATH 202.10 (Term 102)

Quiz 6 (Chap. 8.1-8.3) Duration: 20mn

Name: **ID number:**

Solve the initial value problem

$$X' = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

First, we solve $X' = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 3$

$\Delta = 9 - 12 \Rightarrow \lambda_1 = \frac{3 - i\sqrt{3}}{2}, \lambda_2 = \frac{3 + i\sqrt{3}}{2}$

$(A - \lambda_2 I) K_1 = 0$

$\left(\begin{array}{cc|c} 2 - \frac{3+i\sqrt{3}}{2} & 1 & 0 \\ -1 & 1 - \frac{3+i\sqrt{3}}{2} & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} \frac{1}{2} - \frac{i\sqrt{3}}{2} & 1 & 0 \\ -1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x + y\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = 0$

$y = t$, $K \begin{pmatrix} \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ -1 \end{pmatrix}$

$B_1 = \operatorname{Re} K = \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}, \quad B_2 = \operatorname{Im} K = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$

$X_1 = \left[B_1 \cos \frac{\sqrt{3}}{2} t - B_2 \sin \frac{\sqrt{3}}{2} t\right] e^{3/2 t}$

$$\boxed{X_1 = \left[\begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \cos \frac{\sqrt{3}}{2} t - \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} \sin \frac{\sqrt{3}}{2} t\right] e^{\frac{3}{2} t}}$$

$X_2 = \left[B_2 \cos \frac{\sqrt{3}}{2} t + B_1 \sin \frac{\sqrt{3}}{2} t\right] e^{\frac{3}{2} t}$

$$\boxed{X_2 = \left[\begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} \cos \frac{\sqrt{3}}{2} t + \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} \sin \frac{\sqrt{3}}{2} t\right] e^{\frac{3}{2} t}}$$

$X(t) = c_1 X_1 + c_2 X_2$ is the general solution.

Now, $X(0) = c_1 \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}$

$\frac{c_1}{2} + \frac{\sqrt{3}}{2} c_2 = 1$

$-c_1 = 0 \Rightarrow c_1 = 0$

$c_2 = \frac{2\sqrt{3}}{3}$

$$\Rightarrow \boxed{X(t) = \frac{2\sqrt{3}}{3} \begin{pmatrix} \frac{\sqrt{3}}{2} \cos \frac{\sqrt{3}}{2} t + \frac{1}{2} \sin \frac{\sqrt{3}}{2} t \\ -\sin \frac{\sqrt{3}}{2} t \end{pmatrix} e^{\frac{3}{2} t}}$$