Prob. 1 (12 pts)

Consider the differential equation $x^2y'' + 5xy' + (4 - x^2)y = 0$. The indicial equation has repeated roots $-2, -2$ (no need to check this). Find two linearly independent solutions about $x = 0$.

<u>Prob. 2</u> (10 pts)
Use matrix exponential technique to solve the system

$$x'(t) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix} x(t)$$

<u>Prob.3</u> (12 pts)

Solve the problem $x' = Ax$ where $A = \begin{pmatrix} -1 & -5 \\ 10 & 9 \end{pmatrix}$

Prob.4 (11 pts)

Solve the system $x'(t) = \begin{pmatrix} -1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 2 & -1 \end{pmatrix} x(t)$

Prob.5 (15 pts)

Find a differential equation whose complementary and particular solutions are

$y_c = C_1 e^x + C_2 e^{2x}$ and $y_p = \frac{3}{4} + \frac{1}{2}x - xe^{2x}$, respectively.

Prob. 6 (12 pts)

Consider the initial value problem $y' + \frac{x}{x+5}y = \frac{x^2}{x-1}$, $y(2) = 0$. What is the largest interval on which a unique solution is guaranteed to exist?

(a) $(-5, \infty)$

(b) $(-5, 1)$

(c)$(1, +\infty)$

(d) $[1, +\infty)$

(e) $(-\infty, +\infty)$

Prob. 7 (10 pts)
The solution of the initial value problem

$$\left(\frac{x^2}{y^2} + \frac{3x}{y^4}\right)\frac{dy}{dx} = \frac{2x}{y} + \frac{1}{y^3},\ y(1) = 1$$

is

(a) $2x^2y^2 - y^3 - x = 0$
(b) $2x^2y^2 + y^3 - 3x = 0$
(c) $x^2y^2 + 2y^3 - 3x = 0$
(d) $x^2y^2 - 2y^3 + x = 0$
(e) $x^2y^2 - 3y^3 + 2x = 0$

Prob. 8 (12 pts)
A solution of the problem $-ydx + (x + \sqrt{xy})dy = 0$, $y(1/4) = 1$ satisfies
(a) $y(-1) = e$
(b) $y(1) = e$
(c) $y(\sqrt{2}) = e$
(d) $y(\pi) = e$
(e) $y(e) = e$

Prob. 9 (15 pts)

Consider the differential equation $y'' + 4y = \sec 2t \csc 2t$. The general solution of the corresponding homogeneous equation is $y_c = C_1 \cos 2t + C_2 \sin 2t$ (no need to verify this). A particular solution of the non-homogeneous equation would be

(a) $y_p = -\frac{1}{4} \cos 2t \ln |\sec 2t + \tan 2t| + \frac{1}{4} \sin 2t \ln |\csc 2t - \cot 2t|$
(b) $y_p = -\cos^2 2t \ln |\sec 2t + \tan 2t| + \frac{1}{4} \sin^2 2t \ln |\csc 2t - \cot 2t|$
(c) $y_p = -\frac{1}{4} \cos^2 2t \ln |\csc 2 - \cot 2t| + \frac{1}{4} \sin^2 2t \ln |\sec 2t + \tan 2t|$
(d) $y_p = -\cos 2t \ln |\csc 2t - \cot 2t| + \sin 2t \ln |\sec 2t + \tan 2t|$
(e) $y_p = -\ln |\sec 2t + \tan 2t| + \ln |\csc 2t - \cot 2t|$

<u>Prob. 10</u> (10 pts)

Consider the differential equation $(1-x^2)y'' - 2y' + 3y = 0$. Find the recurrence relation in the process of finding a series solution about $x_0 = 0$.

(a) $a_{n+1} = \frac{-4n-5}{-2(n+1)} a_n$

(b) $a_{n+2} = \frac{n^2-n-3}{(n+2)(n+1)} a_n + \frac{2}{n+2} a_{n+1}$

(c) $a_{n+2} = \frac{-n^2+n+3}{(n+2)(n-1)} a_n - \frac{2}{n+2} a_{n+1}$

(d) $a_{n+2} = \frac{n^2-n-3}{n(n+1)} a_n$

(e) $a_{n+1} = \frac{n^2+n-1}{2(n+1)} a_n$

Prob. 11 (11 pts)
Consider the following non-homogeneous system of differential equations

$$x' = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -2t^{-3} \end{pmatrix}$$

The general solution of the corresponding system is

$$x_c = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} t \\ 2t-1 \end{pmatrix}$$

(No need to verify this). Find a particular solution of the non-homogeneous equation

a) $v(t) = \begin{pmatrix} t^{-1} + \ln t \\ \ln t \end{pmatrix}$

b) $v(t) = \begin{pmatrix} t^{-1} - \ln t \\ t^{-3} + \sqrt{t} \end{pmatrix}$

c) $v(t) = \begin{pmatrix} t^{-1} \\ 2t^{-1} + t^{-2} \end{pmatrix}$

d) $v(t) = \begin{pmatrix} t^{-1} + 2\ln t \\ t^{-3} + \ln t \end{pmatrix}$

e) $v(t) = \begin{pmatrix} t^{-1} + 5\ln t \\ t^{-3} + \ln t \end{pmatrix}$

Prob. 12 (14 pts)

Let $f(x)$ be a function satisfying $f''(x) - 2f'(x) + f(x) = 2e^x$. Which one of the following statement is true

(a) If $f_1(x)$ and $f_2(x)$ are solutions of this equation then $f_1(x) + f_2(x)$ is also a solution of this non-homogeneous equation

(b) This differential equation does not admit any solution

(c) If $f(x) > 0$ for all x then $f'(x) > 0$ for all x as well

(d) If $f_1(x)$ and $f_2(x)$ are solutions of this equation then $\frac{f_1(x)}{f_2(x)}$ must also be a solution of this non-homogeneous equation

(e) If $f'(x) > 0$ for all x then $f(x) > 0$ for all x as well