Major Exam II
Math 202 - Term 112
Key Solution

Prob. 1 (13 points)

Solve the initial value problem on the interval $(-\infty, 0)$

$$4x^2y'' + y = 0; \quad y(-1) = 2, \quad y'(-1) = 4.$$

Solution: (1) Set $t = -x$. Then $x \in (-\infty, 0) \Rightarrow t \in (0, +\infty)$

The D.E becomes $4t^2 \frac{d^2y}{dt^2} + y = 0$ with $\begin{cases} y(+1) = 2 \\ \frac{dy}{dt}(1) = -4 \end{cases}$

2pts

(2) Set $y = t^m$. Then $\frac{d^2y}{dt^2} = m(m-1)t^{m-2}$.

1pt

(3) Substituting in the equation (new equation): We obtain $4m(m-1)t^m + t^m = 0$.

So $(4m(m-1)+1)t^m = 0$.

The auxiliary equation is $4m(m-1)+1 = 0$ 2pts

(4) So $4m^2 - 4m + 1 = 0$ and so $(2m-1)^2 = 0$.

then $m = 1/2$ is a double root. 1pt

(5) the solutions are $y_1 = t^{1/2}$ 1pt and $y_2 = t^{1/2}\ln t$ 1pt, and

the solution is $\boxed{y = y_c = c_1y_1 + c_2y_2 = c_1t^{1/2} + c_2t^{1/2}\ln t}$ 1pt

(6) $\begin{cases} y(1) = 2 \\ \frac{dy}{dt}(1) = -4 \end{cases} \Leftrightarrow \begin{cases} c_1 + 0 = 2 \\ \frac{1}{2}c_1 + c_2 = -4 \end{cases} \Leftrightarrow \begin{cases} c_1 = 2 \text{ 1pt} \\ c_2 = -5 \text{ 1pt} \end{cases}$

(7) the solution of the initial value problem is

$$\boxed{y = 2t^{1/2} - 5t^{1/2}\ln t = 2(-x)^{1/2} - 5(-x)^{1/2}\ln(-x)}$$

2pts

Prob. 2: (13 Points)
Solve the differential equation

$$2y'' - 4y' + 2y = e^x \ln x.$$

Solution:

1. Put the equation in the standard form:

$$y'' - 2y' + y = \frac{1}{2} e^x \ln x \quad 1pt$$

2. the homogeneous equation associated to the DE is:

$$y'' - 2y' + y = 0. \quad 1pt$$

3. The characteristic equation is $r^2 - 2r + 1 = 0$, 1pt
So $(r-1)^2 = 0$. It has 1 as a double root.

③ $y_c = c_1 e^x + c_2 x e^x$ 1pt

④ Set $y_1 = e^x$ and $y_2 = x e^x$ and use Variation of Parameters

Then $W(y_1, y_2) = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^{2x} + x e^{2x} - x e^{2x} = e^{2x} \neq 0.$ 1pt

⑤ $W_1 = \begin{vmatrix} 0 & x e^x \\ \frac{1}{2} e^x \ln x & e^x + x e^x \end{vmatrix} = -\frac{1}{2} x e^{2x} \ln x$ 1pt

$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{2} e^x \ln x \end{vmatrix} = \frac{1}{2} e^{2x} \ln x.$ 1pt

⑥ 1pt $u_1' = -\frac{1}{2} x \ln x = \frac{W_1}{W} \Longrightarrow u_1 = \frac{x^2}{8} - \frac{1}{4} x^2 \ln x$ 1pt

⑦ 1pt $u_2' = \frac{W_2}{W} = \frac{1}{2} \ln x \Rightarrow u_2 = \frac{1}{2} x \ln x - \frac{1}{2} x.$ 1pt

⑧ $y_p = u_1 y_1 + u_2 y_2 = \left(\frac{x^2}{8} - \frac{1}{4} x^2 \ln x\right) e^x + \left(\frac{1}{2} x \ln x - \frac{1}{2} x\right) x e^x$

$= \frac{1}{4} x^2 e^x \ln x - \frac{3}{8} x^2 e^x$ 1pt

⑨ $\boxed{y = y_c + y_p = c_1 e^x + c_2 x e^x + \frac{1}{4} x^2 e^x \ln x - \frac{3}{8} x^2 e^x}$ 1pt

Prob. 3: (16 Points)

(a) Find three linearly independent functions that are annihilated by the differential operator

$$D^3 - 8;\ D = \frac{d}{dx}$$

(b) Use the annihilator approach to solve the differential equation

$$y'' - 9y = 2e^{5x} - 8\cos(2x)$$

(Do not evaluate the constants!)

Solution: $D^3 - 8 = (D-2)(D^2+2D+4)$. 1pt

So the charact. equation would be $(r-2)(r^2+2r+4)=0$.

1pt

So $r = 2$ and $r = -1 \pm i\sqrt{3}$ 1pt

$\Delta = 4 - 16 = -12$

$\Delta = (2i\sqrt{3})^2$.

$r_1 = \frac{-2-2i\sqrt{3}}{2}$, $r_2 = \frac{-2+2i\sqrt{3}}{2}$

$r_1 = -1 - i\sqrt{3}$, $r_2 = -1+i\sqrt{3}$

Therefore the required linearly independent functions are,

e^{2x} 1pt, $e^{-x}\cos(\sqrt{3}x)$ 2pt, $e^{-x}\sin(\sqrt{3}x)$ 2pt

(b) Annihilator approach to solve $y'' - 9y = 2e^{5x} - 8\cos(2x)$

1pt. Annihilator for e^{5x} is $(D-5)$

1pt. Annihilator for $\cos 2x$ is (D^2+4)

$\Rightarrow$ An annihilator for $2e^{5x} - 8\cos(2x)$ is $(D-5)(D^2+4)$. 2 pts. Then we obtain

2pts $(D^2-9)(D-5)(D^2+4)(y)=0$. The corresponding "char. Equation" is $(r^2-9)(r-5)(r^2+4)=0$ and the roots are $r = 3, -3, 5, 2i, -2i$. Thus the general solution is

2pts $\boxed{y = C_1 e^{3x} + C_2 e^{-3x} + C_3 e^{5x} + C_4\cos(2x) + C_5\sin(2x)}$

Prob. 4: (14 Points)
Solve the initial value problem

$$y''' - 5y'' + 100y' - 500y = 0; \; y(0) = 0, \; y'(0) = 10, \; y''(0) = 250$$

given that $y_1(x) = e^{5x}$ is a solution of the differential equation.

2 pts

Solution: ① The charact. equation is $r^3 - 5r^2 + 100r - 500 = 0$

② Since e^{5x} is a solution for the D.E., $r = 5$ must be a root of the "charact. Equation".

③ By Long division $r^3 - 5r^2 + 100r - 500 = \underbrace{(r-5)(r^2+100)}_{1pt}$

and so the roots are $r = 5$, $r = \pm 10i$
1pt 1pt

④ the general solution is given by

$$\boxed{y = c_1 e^{5x} + c_2 \cos(10x) + c_3 \sin(10x)}$$ 3pts

(1) (1) (1)

⑤ $y' = 5c_1 e^{5x} - 10c_2 \sin(10x) + 10c_3 \cos(10x)$ 1pt

$y'' = 25c_1 e^{5x} - 100c_2 \cos(10x) - 100c_3 \sin(10x)$ 1pt

$$\begin{cases} y(0)=0 \\ y'(0)=10 \\ y''(0)=250 \end{cases} \Longleftrightarrow \begin{cases} c_1 + c_2 = 0 \\ 5c_1 + 10c_3 = 10 \\ 25c_1 - 100c_2 = 250 \end{cases} \Longleftrightarrow \begin{cases} c_1 + c_2 = 0 & (1) \\ c_1 + 2c_3 = 2 & (2) \\ c_1 - 4c_2 = 10 & (3) \end{cases}$$

$(3) - (1) \Rightarrow -5c_2 = 10$. so $\boxed{c_2 = -2}$ 1pt

$(1) \Longrightarrow \boxed{c_1 = -c_2 = 2}$ 1pt $\quad (2) \Longrightarrow 2c_3 = 2 - c_1 = 0$

$\boxed{c_3 = 0}$ 1pt

⑥ the solution is $\boxed{y = 2e^{5x} - 2\cos(10x)}$ 1pt

Prob. 5: (11 Points)

Let $y_1 = x^{-1/2}\sin x$ be a solution of $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$. Use the reduction of order method to find a second solution.

Write given D.E. in Standard form

$$y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right) = 0 \quad , \quad x \neq 0$$

(3) pts

The second solution is given by the reduction of order method

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{(y_1(x))^2}\,dx$$

(4) pts

$$= x^{-\frac{1}{2}} \sin x \int \frac{e^{-\int \frac{1}{x}dx}}{\left(x^{-1/2}\sin x\right)^2}\,dx$$

$$= x^{-\frac{1}{2}} \sin x \int \frac{e^{-\ln x}}{x^{-1}\sin^2 x}\,dx$$

$$= x^{-\frac{1}{2}} \sin x \int \cancel{\frac{e^{\ln x^{-1}}}{x^{-1}\sin^2 x}}\,dx$$

(2) pts

$$= x^{-\frac{1}{2}} \sin x \int \operatorname{cosec}^2 x\,dx$$

$$= -x^{-\frac{1}{2}} \sin x \cot x = -x^{-\frac{1}{2}} \sin x \frac{\cos x}{\sin x}$$

(2) pts

$$= -x^{-\frac{1}{2}} \cos x$$

Prob. 6: (11 Points)
Consider the differential equation

$$y'' + y = \sec x + e^x$$

(a) Check that $x \sin x + (\cos x)\ln(\cos x)$ is a particular solution of

$$y'' + y = \sec x.$$

(b) Find the general solution of $y'' + y = \sec x + e^x$.

(a) Let $y_{p_1} = x \sin x + \cos x \ln(\cos x)$ $\quad (\cos x > 0)$ $\quad$ (2) pts

Then $y'_{p_1} = \cancel{\sin x} + x\cos x + \cancel{\cos x}\dfrac{(-\sin x)}{\cancel{\cos x}} - \ln(\cos x)\sin x$

$y''_{p_1} = \cos x + x(-\sin x) - \cos x \ln(\cos x) + \dfrac{\sin^2 x}{\cos x}$ $\quad$ (2) pts

So $y''_{p_1} + y_{p_1} = \cos x - x\cancel{\sin x} - \cancel{\cos x \ln(\cos x)} + \dfrac{\sin^2 x}{\cos x} + x\cancel{\sin x} + \cancel{\cos x \ln(\cos x)}$

$= \cos x + \dfrac{\sin^2 x}{\cos x} = \dfrac{\cos^2 x + \sin^2 x}{\cos x} = \sec x$ $\quad$ (1) pt

(b) If we look for a particular solution of $y'' + y = e^x$ in the form $y_{p_2} = c e^x$, then we have

$c e^x + c e^x = e^x \Rightarrow 2ce^x = e^x \Rightarrow c = \frac{1}{2}$.

So $y_{p_2} = \frac{1}{2} e^x$. $\quad$ (3) pts

The general solution of the homogeneous equation $y'' + y = 0$ is $y_g = c_1 \cos x + c_2 \sin x$. $\quad$ (1) pt

Hence by superposition principle, we have

$y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\cos x) + \frac{1}{2} e^x$. $\quad$ (2) pts

Prob. 7: (11 Points)

Show that x, $x\ln x$ and x^2 form a fundamental set of solutions (are solutions and are linearly independent) of the differential equation

$$x^3y''' - x^2y'' + 2xy' - 2y = 0, \ x > 0.$$

First we verify that $y_1 = x$, $y_2 = x\ln x$ and $y_3 = x^2$ are solutions of the differential equation (D.E.).

$y_1 = x$: D.E. $\Rightarrow x^3(0) - x^2(0) + 2x(1) - 2x = 0$ (2) pts

$y_2 = x\ln x$: $y_2' = \ln x + 1$, $y_2'' = \frac{1}{x}$, $y_2''' = -\frac{1}{x^2}$

D.E. $\Rightarrow x^3\left(-\frac{1}{x^2}\right) - x^2\left(\frac{1}{x}\right) + 2x(\ln x + 1) - 2x\ln x = 0$ (2) pts

$y_3 = x^2$: $y_3' = 2x$, $y_3'' = 2$, $y_3''' = 0$

D.E. $\Rightarrow x^3(0) - x^2(2) + 2x(2x) - 2x^2 = 0$ (2) pts

To check linear independence of the functions, we calculate their Wronskian W:

$$W = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$ (2) pts

$= x\left[2\ln x + \cancel{2} - \cancel{2}\right] - 1\left[2x\ln x - x\right] + 0$ (2) pts

$= \cancel{2x\ln x} + x - \cancel{2x\ln x} = x \neq 0$ in $(0, \infty)$. (1) pt

Thus x, $x\ln x$ and x^2 form a fundamental set of solutions.

Prob. 8: (11 Points)
Let $y = C_1 \cos \omega x + C_2 \sin \omega x$, $\omega \neq 1$, be a 2-parameter family of solutions of the differential equation $y'' + \omega^2 y = 0$. Determine whether a member of the family can be found that satisfies the boundary conditions $y(0) = 1$ and $y'(\frac{\pi}{2\omega}) = -1$.

Suppose a member of the family can be found that satisfies the boundary conditions. (2) pts

$y = c_1 \cos \omega x + C_2 \sin \omega x \quad , \omega \neq 1$

$y' = -c_1 \omega \sin \omega x + c_2 \omega \cos \omega x$ (2) pts

$y(0) = 1 \Rightarrow \quad 1 = c_1 + c_2(0) \Rightarrow c_1 = 1$ (3) pts

$y'(\frac{\pi}{2\omega}) = -1 \Rightarrow \quad -1 = -c_1 \omega \sin(\omega \frac{\pi}{2\omega}) + c_2 \omega \cos(\frac{\omega \pi}{2\omega})$

$-1 = -c_1 \omega$

$c_1 = \frac{1}{\omega}$ where $\omega \neq 1$ (3) pts

This contradics $c_1 = 1$. (1) pt

So the given boundary-value problem has no solution.