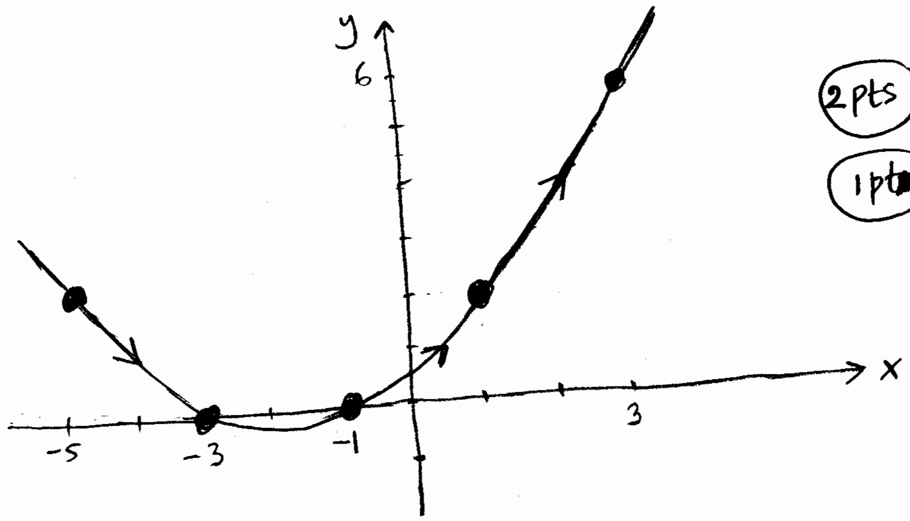
Q1. (a) Sketch the parametric curve **C:** $x = 2t - 1,\ y = t + t^2$. Indicate by an arrow how the graph is traced as t increases. **[5pts]**

Solution:

t	-2	-1	0	1	2
x	-5	-3	-1	1	3
y	2	0	0	2	6

2 pts



2 pts correct graph

1 pt arrow

(b) Find an equation of the tangent line to the curve **C** at the point $(1, 2)$. **[5pts]**

Solution: $(x_0, y_0) = (1, 2) \Longrightarrow t_0 = 1$ 1pt

The slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{t=t_0} = \frac{1 + 2t_0}{2} = \frac{3}{2}$$ 2pts

An equation for the tangent line is

$$y - 2 = m(x-1) = \frac{3}{2}(x-1)$$

$$y = \frac{3}{2}x + \frac{1}{2}$$

2 pts

Q2. Find the surface area obtained by rotating the curve $x = \cos^2 t,\ y = \sin^2 t$, $0 \le t \le \pi/2$ about the y-axis. **[9pts]**

Solution:

3 $$S = \int_0^{\pi/2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2 $$= \int_0^{\pi/2} 2\pi \cos^2 t \sqrt{(-2\sin t\cos t)^2 + (2\sin t \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} 2\pi \cos^2 t \sqrt{8 \sin^2 t \cos^2 t}\, dt$$

1 $$= 4\sqrt{2}\,\pi \int_0^{\pi/2} \cos^3 t \sin t \, dt$$

1 $$= 4\sqrt{2}\,\pi \left[-\frac{\cos^4 t}{4}\right]_0^{\pi/2}$$

1 $$= \sqrt{2}\,\pi$$

Q3. (a) Find all values of c such that vectors $\vec{v} = \langle c, 5, 2\rangle$ and $\vec{w} = \langle 3c, c, -1\rangle$ are orthogonal. **[5pts]**

Solution: $\vec{v}$ and $\vec{w}$ are orthogonal

$\iff \vec{u}\cdot\vec{w} = 0$ 2 pts

$\iff 3c^2 + 5c - 2 = 0$ 2 pts

$\iff (3c-1)(c+2) = 0$

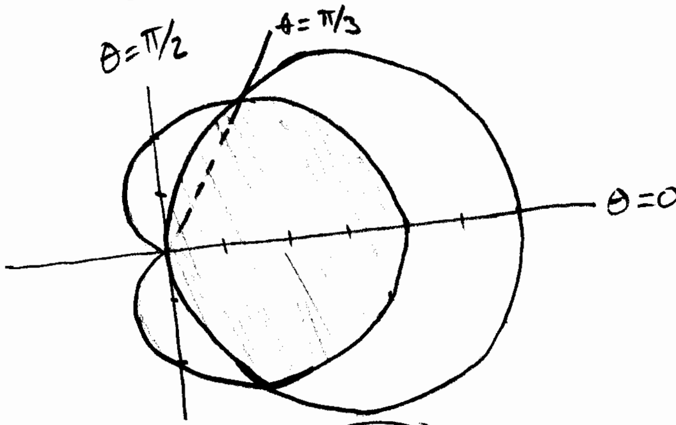
$c = \frac{1}{3}$ or $c = -2$ 1pt

(b) Find the direction cosines of the vector $\vec{u} = \langle 2, 1, -2\rangle$. **[4pts]**

Solution: $|\vec{u}| = \sqrt{4+1+4} = 3$ 1pt

$$\left.\begin{aligned} \cos\alpha &= \frac{2}{3} \\ \cos\beta &= \frac{1}{3} \\ \cos\gamma &= -\frac{2}{3} \end{aligned}\right\} \quad 1 + 1 + 1$$

Q4. Find the area of the region that lies inside both the polar curves $r = 2 + 2\cos\theta$ and $r = 6\cos\theta$. **[13pts]**



2 pts

Solution:

Intersection:

2 pts

$$\begin{cases} 2 + 2\cos\theta = 6\cos\theta \\ \Rightarrow \quad \cos\theta = \frac{1}{2} \\ \Rightarrow \quad \theta = \pi/3 \end{cases}$$

$$A = 2(A_1 + A_2)$$

2 pts

$$A_1 = \int_0^{\pi/3} \frac{1}{2}(2 + 2\cos\theta)^2 \, d\theta$$

2 pts

$$= 2\int_0^{\pi/3} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= 2\int_0^{\pi/3} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3}$$

$$= 2\left[\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}\right] = \pi + \frac{9\sqrt{3}}{4}$$

2 pts

$$A_2 = \int_{\pi/3}^{\pi/2} \frac{1}{2}(36\cos^2\theta) \, d\theta = \int_{\pi/3}^{\pi/2} 9(1 + \cos 2\theta) \, d\theta$$

$$= 9\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} = 9\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{3\pi}{2} - \frac{9\sqrt{3}}{4}$$

2 pts

Thus, the required area is

$$A = 2\left[\pi + \frac{9\sqrt{3}}{4} + \frac{3\pi}{2} - \frac{9\sqrt{3}}{4}\right]$$

$$= 5\pi$$

1 pt

Q5. (a) Given the points $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,-1)$ and $D(3,m,0)$, find the volume of the parallelepiped with adjacent edges AB, AC, AD. **[7pts]**

Solution: $\vec{AB} = \langle 1,-1,3 \rangle$, $\vec{AC} = \langle 2,-3,0 \rangle$, $\vec{AD} = \langle 1, m-1, 1 \rangle$ 3 pts

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 0 \\ 1 & m-1 & 1 \end{vmatrix} = 6m+2$$ 3 pts

The volume is

$$V = \left| \vec{AB} \cdot (\vec{AC} \times \vec{AD}) \right|$$

$$= |6m+2|$$ 1 pt

(b) Find all values of m such that the volume in (a) is 4. **[3pts]**

Solution:

$$V = |6m+2| = 4$$ 1 pt

$6m+2 = 4$ or $6m+2 = -4$

$\Rightarrow m = \frac{1}{3}$ 1 pt

$\Rightarrow m = -1$ 1 pt