1. **Problem 1.**(10 marks)

   Given (u_0, u_1) in $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, where Ω is a bounded and regular domain of $\mathbb{R}^n$. Prove the existence of a weak solution of the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_\Omega u(x,t)dx = 0, & \text{in } \Omega,\ t > 0 \\ u = 0, & \text{on } \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

Problem 2. (10 marks)

Given the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = f(x,t), & \text{in } \Omega,\ t > 0 \\ u = 0, & \text{on } \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

a) Set the problem in the form $U_t + AU = F$, for $A : V \to V'$ to be determined and F and V to be determined too.

b) Check if A is monotone, hemicontinuous, and bounded.

Problem 3 (10 marks)

Given u_0 in $L^2(\mathbb{R}^n)$. Prove the existence of a weak solution of the problem

$$\begin{cases} u_t - div(|\nabla u|^{p-1}\nabla u) + |u|^{p-1}u = 0, & \text{in } \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n \end{cases}$$

Do not check Hemicontinuity.

Problem 4 (10 marks)

Suppose that Ω is a bounded regular domain of $\mathbb{R}^n$. Given u_0 in $L^2(\Omega)$ and f in $L^2\left((0,T);H^{-1}(\Omega)\right)$. Show that

$$\begin{cases} u_t - \Delta u = f, & \text{in } \Omega,\ t > 0 \\ u = 0, & \text{on } \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x), & \text{in } \Omega \end{cases}$$

has a unique weak solution

$$u \in L^2((0,T);H_0^1(\Omega)) \cap C([0,T);L^2(\Omega))$$