1. **Problem 1**.(4 marks)

   Let H be a Hilbert space and $A : H \to H$ be a linear and an isomorphism self-adjoint operator. Show that A^{-1} is self-adjoint..

Problem 2. (6 marks)

Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a linear maximal operator such that, for some $c_0 > 0$, $(Av, v) \geq c_0||v||^2$.

a) Give a condition on β so that $B = A + \beta I$ is monotone.

b) If $c_0 < 1$, show that B is maximal.

Problem 3 (8 marks)

Given u_0 in $L^2(\mathbb{R}^n)$. Solve

$$\begin{cases} u_t - \Delta u + u = 0, & \text{in } \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n \end{cases}$$

Problem 4 (12 marks)

Suppose that Ω is a bounded regular domain of $\mathbb{R}^n$. Given u_0, v_0 in $L^2(\Omega)$. Show that

$$\begin{cases} u_t - \Delta u + u - \frac{1}{2}v = 0, & \text{in } \Omega,\ t > 0 \\ v_t - \Delta v - \frac{1}{2}u + v = 0, & \text{in } \Omega,\ t > 0 \\ u = v = 0, & \text{on } \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \text{in } \Omega \end{cases}$$

has a unique solution (u, v) such that

$$u, v \in C((0,+\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0,+\infty); L^2(\Omega)) \cap C^1((0,+\infty); L^2(\Omega))$$