

HOMEWORK – 5
 Due Saturday 3-12-2011

1a) Suppose A is a real symmetric 805×805 matrix with eigenvalues 1.00, 1.01, 1.02, ..., 8.98, 8.99, 9.00 and also 10, 12, 16, 24. How many steps of the conjugate gradient iteration must you take to be sure of reducing the initial error $\|e_0\|_A$ by a factor of 10^6 ?

1b) Suppose A is a dense symmetric positive definite 1000×1000 matrix with $k(A) = 100$. Estimate roughly how many flops are required to solve $Ax=b$ to ten-digit accuracy by Conjugate Gradient Method.

2) Consider the recurrence

$$\gamma_{j+1} \mathbf{v}^{(j+1)} = A\mathbf{v}^{(j)} - \delta_j \mathbf{v}^{(j)} - \gamma_j \mathbf{v}^{(j-1)}, \quad 1 \leq j \leq k,$$

where $\mathbf{v}^{(1)}$ is arbitrary with $\|\mathbf{v}^{(1)}\| = 1$, $\mathbf{v}^{(0)} = 0$, $\delta_j = (A\mathbf{v}^{(j)}, \mathbf{v}^{(j)})$, and γ_j is chosen so that $\|\mathbf{v}^{(j)}\| = 1$. Prove that this procedure generates an orthonormal basis for the Krylov subspace $\mathcal{K}_k(A, \mathbf{v}^{(1)})$. (Hint: use induction and note that for a symmetric matrix A

$$\langle A\mathbf{v}^{(j)}, \mathbf{v}^{(j-1)} \rangle = \langle \mathbf{v}^{(j)}, A\mathbf{v}^{(j-1)} \rangle,$$

and also that $A\mathbf{v}^{(j-1)} = \gamma_j \mathbf{v}^{(j)} + \mathbf{w}$, with $\mathbf{w} \in \text{span}\{\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(j-1)}\}$.)

3) Let $\{r_k\}$ and $\{p_k\}$ be generated by the Conjugate Gradient method. Prove that:

- (a) $r_k^T p_j = 0, \quad j = 0, 1, \dots, k-1$
- (b) $r_k^T r_j = 0, \quad j = 0, 1, \dots, k-1$
- (c) $p_j^T A p_k = 0, \quad j = 0, 1, \dots, k-1$

4) The following boundary value problem is to be solved computationally

$$\begin{aligned} u_{xx} + u_{yy} &= -2 & (x, y) \in \Omega = (0,1) \times (0,1) \\ u &= 0, & (x, y) \text{ on AB, BC, and AD} \\ u &= 1, & (x, y) \text{ on DC} \end{aligned}$$

Where $\Omega \in R^2$ is the square domain $(0,1) \times (0,1)$ shown in the figure

For the computation the indicated uniform grid with step size $h = 1/1001$ is introduced.

At the 1,000,000 numbered gridpoints $p = 1, 2, \dots, 1000000$ let u_p denote (unknown) approximations of the values of the solution u . (Note that unnumbered gridpoints belong to boundary segments where the function values are prescribed).

At each numbered node p the differential equation is approximated by a linear equation as follows:

(*) For any interior node p of Ω use $4u_p - u_n - u_w - u_s - u_e - 2h^2 = 0$ where u_n, u_w, u_s, u_e are the unknown values at the node 'north', 'west', 'south', and 'east' of p , respectively.

This results in a linear system of equations with size 1,000,000x1,000,000 matrix that turns out to be symmetric, positive definite.

- Write a program that solve this linear system by 1) CG 2) MINRES 3)GMRES 4)GS. (use: $\text{tol}=10^{-10}$)
- Produce a plot with four curves on it; the residual norms $\|r_2\|_2$ for 1) CG 2) MINRES 3)GMRES 4)GS and the estimate $2(\sqrt{\kappa}-1)^n / (\sqrt{\kappa}+1)^n$.
- Produce a table showing the cpu time for the four methods.
- Approximate the value of u at $(x, y) = (0.5, 0.5)$.
- Comment on your results.

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	1	2	999	1000