King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 590 Exam I , Semester I, (111)
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Name:	
ID:	

Q		Points
1		30
2		30
3		30
4(a,b)		20
5		20
6		20
BONUS		10
Total		**150**

☺ **Say Bismillah & Good luck** ☺

(1) Determine SVD of the matrix A (show all your work)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad U = \qquad \Sigma = \qquad V =$$

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(2) The singular value decomposition of B is

$$B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .87 & .39 & -.30 \\ .30 & -.91 & -.30 \\ .39 & -.17 & .90 \end{bmatrix} \begin{bmatrix} 2.49 & 0 \\ 0 & 1.33 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} .23 & .97 \\ .97 & -.23 \end{bmatrix}$$

Express B as the sum of rank-one matrices:

$$B = \begin{bmatrix} & \\ & \\ & \end{bmatrix} + \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

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(3) The QR factorization of A is:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7.71 & 0.5774 & 0.4082 \\ 0.7071 & -0.5774 & -0.4082 \\ 0 & 0.5774 & -0.8165 \end{bmatrix} \begin{bmatrix} 1.4142 & 2.8284 \\ 0 & 1.7321 \\ 0 & 0 \end{bmatrix}$$

Solve the least square problem Ax = b where $b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$

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(4) Let $A \in R^{n\times n}, m > n$

a) Explain why the rank of A is exactly the number of its nonzero singular values

b) If $A = U\Sigma V^T$ is a singular value decomposition, what are the singular value decomposition of A^T, A^+

[Note that $A^+ \in R^{n\times m}$, $A^+ = (A^T A)^{-1} A^T$]

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(5) Let $A \in R^{m\times n}, m > n$ be such that $\|Ax\|_2 = \|x\|_2$ for all $x \in R^n$. Show that A has orthonormal columns; that is, that $A^T A = I_n$.

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(6) Suppose the matrix $A \in R^{2m\times 2n}$ has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where $A_1, A_2 \in R^{m\times n}$ are nonzero matrices. Show that

$$\frac{\|A\|_2}{\|A_1\|_F} \geq \frac{\|A_2\|_2}{\|A\|_F}$$

[Hint: use SVD]

BONUS:

Given $A \in R^{m\times n}, m > n$ and a nonzero vector $v \in R^m$. The following algorithm overwrites A with HA where $P = I - 2vv^T / v^T v$

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function [A] = row.house(A,v)
    β = −2 / v^T v
    w = β A^T v
    A = A + v w^T
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Count how many flops are required for this algorithm. Write your answer in terms n and m.
[A **flop** is a floating point operation. Each addition, subtraction, multiplication, division or square root counts as one flop.]