King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 535 [Functional Analysis I]
First Semester 2011-2012 (111)

Final Exam: January 7, 2012 Time: 3 Hours

1. (a) Every metric on a vector space is not necessarily a norm. Justify this statement by means of a suitable example.

   (b) Define a contraction on a metric space. Let $T : [1, \infty) \to [1, \infty)$ be given by $Tx = \frac{25}{26}\left(x + \frac{1}{x}\right)$. Use Banach fixed point theorem to find a unique fixed point of T.

2. (a) Let p be a fixed integer such that $1 \leq p \leq \infty$. Define $l_p = \left\{ \{x = \{x_n\} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$. Consider l_p under its usual sum norm and show that the dual of l_p is l_q where $\frac{1}{p} + \frac{1}{q} = 1$.

   (b) Prove that a closed linear mapping of a Banach space E into a Banach space F is continuous.

3. (a) Let $\{\alpha_n\}$ be a sequence of reals. Define a sequence of functionals on l_1 (under its usual norm) by

   $$f_n(x) = \sum_{k=1}^{n} \alpha_k \xi_k, \quad x = \{f_k\} \in l_1.$$

   Show that each f_n is linear and continuous and $\|f_n\| = \max_{1 \leq k \leq n} |\alpha_k|$. Assume that $\sum_{k=1}^{\infty} \alpha_k \xi_k$ is convergent for every $\{\xi_k\} \in l_1$. Use uniform boundedness principle, to show that $\{\alpha_n\}$ is bounded.

   (b) Let $x \neq 0$ be any element of a normed space X. Prove that there exists a bounded linear functional f on X such that $\|f\| = 1$ and $f(x) = \|x\|$. Hence show that $\|x\| = \sup_{0 \neq f \in X^*} \frac{|f(x)|}{\|f\|}$ where X^* denotes dual of X.

4. (a) Let f be a continuous linear functional on a Hilbert space H. Prove that there exists a unique $z \in H$ such that $f(x) =< x, z >$ for all $x \in H$ and $\|f\| = \|z\|$.

   (b) Let M be a nonempty subset of a Hilbert space H. If the span of M is dense in H, then show that $M^{\perp} = \{0\}$.

5. (a) Let E be a normed space and suppose that E^* is separable. Then prove that E is separable.

(b) Let $\{x_n\}$ be a sequence in a normed space X. If $\{x_n\}$ converges weakly to $x \in X$, then show that:
(i) the sequence $\{\|x_n\|\}$ is bounded.
(ii) for all $f \in M^*$, we have $f(x_n) \to f(x)$ where M^* is a strongly dense subset of X^*.

6. (a) Prove that a bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.

(b) Let A be a complex Banach algebra with identity e. Then prove that the set of all invertible elements of A is an open set. If $x \in A$, then is it true that the spectrum of $x, \sigma(x)$, is an empty set?