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MAJOR No. 1
MATH. 533-111

Prob. 1

Suppose that $z(t) = x(t) + iy(t)$ is a complex, differentiable function of a real parameter $t \in (a, b)$ that does not vanish in (a, b). Show that

(i) $\frac{d}{dt} \arg z(t) = (xy' - x'y) |z(t)|^{-2}$

(ii) $\frac{d}{dt} |z(t)| = (xy' + x'y) |z(t)|^{-1}$.

Prob. 2

Explain the geometric meaning of

(i) $\{z : 0 < \arg \frac{z+i}{z-i} < \pi\}$

(ii) $\{z : |z| + \mathbf{R}z \leq 1\}$.

Prob. 3

We denote by $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. If f is analytic and does not vanish in a domain D, prove that

$$\Delta |f(z)| = |f(z)|^{-1} |f'(z)|^2, \ z \in D.$$

Prob. 4

Suppose that $w = f(z)$ is analytic in a domain D and $f(D) \cap (-\infty, 0] = \emptyset$. Show that $F(z) = \log |f(z)| + iArgf(z)$ is analytic. Evaluate F'.

Prob. 5

Compute $\sum_{k=1}^{n} ke^{ikt}$. Study the convergence of $z_n = \frac{1}{n} \sum_{k=1}^{n} ke^{ikt}$ and $w_n = \frac{1}{n^2} \sum_{k=1}^{n} ke^{ikt}$.

Prob. 6

Find the radius of convergence of

(a) $\sum_{n=0}^{\infty} n^{(-1)^n} z^n$

(b) $\sum_{n=0}^{\infty} z^{n!}$

Prob. 7

At which points are the following functions differentiable?

(a) $f(z) = x^2 + y^2 + 2ixy$

(b) $f(z) = z\mathbf{R}z$.

Prob. 8

Let f be a continuous function in an open set $\Omega \subset \mathbf{C}$ and assume that f^2 is holomorphic in Ω. Assume moreover that f does not vanish in Ω. Show that f is holomorphic in Ω.

Prob. 9

Let

$$f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2}, \ z \neq 0 \\ 0, \ z = 0. \end{cases}$$

(a) Show that f is continuous at 0.

(b) Show that $\lim_{z\to 0} \frac{f(z)-f(0)}{z}$ exists along any fixed direction, that all these limits are equal to 0, but that f is not differentiable at 0.

(c) Show that the Cauchy-Riemann equations hold at the origin.