King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 302 Exam I

Semester (111) October 13, 2011 Time: 1:00 - 2:30 pm

Name: ....

I.D: .... Section: ....

Problem	Points
1	____ / 10
2	____ / 10
3	____ / 15
4	____ / 15
Total	____ / 50

Exercise 1. Let $S = \{(a, 4a, b, a+b) \mid a, b \in \mathbb{R}\}$.
(*i*) Show that S is a subspace of $\mathbb{R}^4$.
(*ii*) Find a basis of S and evaluate $\dim(S)$.

Solution.

(i) • $0_{\mathbb{R}^4} = (0,0,0,0) = (0, 4\times 0, 0, 0+0) \Rightarrow 0_{\mathbb{R}^4} \in S$.

• Let $U, V \in S$. Then there exist $a, b, a_1, b_1 \in \mathbb{R}$ such that

$U = (a, 4a, b, a+b)$, $V = (a_1, 4a_1, b_1, a_1+b_1)$.

Hence $U+V = (a+a_1, 4(a+a_1), b+b_1, (a+a_1)+(b+b_1))$

$\Rightarrow U+V \in S$.

• Let $U = (a, 4a, b, a+b) \in S$ and $\alpha \in \mathbb{R}$.

Then $\alpha U = (\alpha a, 4(\alpha a), \alpha b, (\alpha a)+(\alpha b)) \Rightarrow \alpha . U \in S$

It follows that S is a subspace of $\mathbb{R}^4$.

(ii) Let $U = (a, 4a, b, a+b) \in S$.

Then $U = (a, 4a, 0, a) + (0, 0, b, b)$

$= a.(1, 4, 0, 1) + b(0, 0, 1, 1)$

This implies that the vectors $U_1 = (1, 4, 0, 1)$, $U_2 = (0, 0, 1, 1)$ span the subspace S.

It remains to show that U_1, U_2 are linearly independent.

Indeed, if $\alpha, \beta \in \mathbb{R}$ and $\alpha U_1 + \beta U_2 = (0,0,0,0)$, then

$(\alpha, 4\alpha, \beta, \alpha+\beta) = (0,0,0,0)$.

This leads to $\alpha = \beta = 0$

le conclude that $\{U_1, U_2\}$ is a basis of S.

insequently, $\dim(S) = 2$.

Exercise 2. Consider the following two matrices:

$$A = \begin{pmatrix} 1 & -2 & 5 \\ 4 & -5 & 8 \\ -3 & 3 & -3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(*a*) Find the reduced row-echelon form of the augmented matrix $[A \mid B]$.

(*b*) Is the system $AX = B$ is consistent (has a solution)? Justify your answer.

(*c*) Find the dimension of the solution space of the homogeneous system

$$AX = O.$$

Solution.

(a) $$[A \mid B] = \left[\begin{array}{ccc|c} 1 & -2 & 5 & 2 \\ 4 & -5 & 8 & 1 \\ -3 & 3 & -3 & 1 \end{array}\right] \xrightarrow[R_3 + 3R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 5 & 2 \\ 0 & 3 & -12 & -7 \\ 0 & -3 & 12 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 5 & 2 \\ 0 & 3 & -12 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\left(\frac{1}{3}\right) R_2} \left[\begin{array}{ccc|c} 1 & -2 & 5 & 2 \\ 0 & 1 & -4 & -\frac{7}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -\frac{8}{3} \\ 0 & 1 & -4 & -\frac{7}{3} \\ 0 & 0 & 0 & 1 \end{array}\right] \cancel{= [A \mid C], \text{ where } C =}$$

(b) As $\operatorname{rank}([A \mid B]) = \operatorname{rank}(A) = 2$, the system $AX = B$ is consistent.

(c). The dimension of the solution space of the homogeneous system $AX = O$ is given by

$$d = 3 - \operatorname{rank}(A) = 3 - 2 = 1$$

Exercise 3. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix}.$$

(*i*) Use Gauss-Jordan Method to find the inverse of A.

(*ii*) Solve the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 9x_2 + 3x_3 = -1 \\ x_1 + 4x_3 = 9 \end{cases}$$

Solution. $\boxed{(i)}$

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 9 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_{1,3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 2 & 9 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 9 & -5 & 0 & 1 & -2 \\ 0 & 2 & -1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{\left(\frac{1}{2}\right)R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 9 & -5 & 0 & 1 & -2 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_{2,3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 9 & -5 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_3-9R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{9}{2} & 1 & \frac{5}{2} \end{array}\right]$$

$$\xrightarrow{(-2)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 9 & -2 & -5 \end{array}\right] \xrightarrow[R_2+\frac{1}{2}R_3]{R_1-4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -36 & 8 & 21 \\ 0 & 1 & 0 & 5 & -1 & -3 \\ 0 & 0 & 1 & 9 & -2 & -5 \end{array}\right]$$

Thus, A is invertible and $A^{-1} = \begin{pmatrix} -36 & 8 & 21 \\ 5 & -1 & -3 \\ 9 & -2 & -5 \end{pmatrix}$

$\boxed{(ii)}$ The matrix form of the system is:

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 9 \end{pmatrix}$. As A is invertible, the system has a unique solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 5 \\ -1 \\ 9 \end{pmatrix}$,

$$\Rightarrow \begin{cases} x = -180 - 8 + 189 = 1 \\ y = 25 + 1 - 27 = -1 \\ z = 45 + 2 - 45 = 2 \end{cases}$$

Exercise 4. Let k be a real number and

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ k & -2 & 0 \end{pmatrix}.$$

(*a*) Find all values of k for which A has a repeated eigenvalue.
(*b*) Suppose that $k = 1$.
- Find all eigenvalues of A and order them $\lambda_1 \leq \lambda_2 \leq \lambda_3$.
- Find an eigenvector of A corresponding to λ_2.

Solution.

(a) $\det(\lambda I_3 - A) = \begin{vmatrix} \lambda+1 & -1 & -1 \\ 0 & \lambda-1 & 1 \\ -k & 2 & \lambda \end{vmatrix}$

$= (\lambda+1)\begin{vmatrix} \lambda-1 & 1 \\ 2 & \lambda \end{vmatrix} + (-k)\begin{vmatrix} -1 & -1 \\ \lambda-1 & 1 \end{vmatrix}$

$= (\lambda+1)(\lambda^2-\lambda-2) - k(-1+\lambda-1)$

$= (\lambda+1)(\lambda-2)(\lambda+1) - k(\lambda-2)$

$= (\lambda-2)((\lambda+1)^2 - k) = (\lambda-2)(\lambda+1-\sqrt{k})(\lambda+1-\sqrt{k})$

The eigenvalues of A are $2, -1+\sqrt{k}, -1-\sqrt{k}$.

- The eigenvalue 2 is repeated if and only if 2 is a zero of $(\lambda+1)^2-k$; this means $(2+1)^2-k=0$; which is equivalent to $\underline{k=9}$

- Now, A has a repeated eigenvalue distinct from 2 if and only if $-1+\sqrt{k} = -1-\sqrt{k}$; that is to say $k=0$

We conclude that A has repeated eigenvalues iff $\underline{k=0}$ or $\underline{k=9}$

(b) For $k=1$, the eigenvalues of A are $2, -1+1, -1-1$

$\lambda_1 = -2 < \lambda_2 = 0 < \lambda_3 = 2$

Eigenvectors of A corresponding to $\lambda = 0$

We solve the homogeneous system $(A - 0I_3)X = 0$, with $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{cases} -x + y + z = 0 \\ y - z = 0 \\ x - 2y = 0 \end{cases} \iff \begin{cases} x = 2y \\ z = y \end{cases}$$

Hence $X = \begin{pmatrix} 2y \\ y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Therefore, $E = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector associated with $\lambda_2 = 0$.