King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Semester (111) December 19, 2011

Math 302
Quiz 5

Name: ....

ID: ....

Exercise 1. Let $f(z) = zRe(z)$. Show that Cauchy-Riemann Equations do not hold at any point except at $z = 0$. Is f differentiable at $z_0 = 0$?

Solution. Let $z = x + iy$; then $f(z) = x^2 + ixy$. Set $u(x, y) = x^2$ and $v(x, y) = xy$. Then

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2x, \dfrac{\partial v}{\partial y} = x \\ \\ \dfrac{\partial u}{\partial y} = 0, \quad \dfrac{\partial v}{\partial x} = y \end{cases}$$

Hence CRE are satisfied iff $x = y = 0 \Leftrightarrow z = 0$.

Differentiability at 0.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h\text{Re}(h)}{h} = \lim_{h\to 0} \text{Re}(h) = 0$$

Thus $f'(0) = 0$. □

Exercise 2. Let $u(x, y) = 2x - 2xy$.

(1) Show that u is a harmonic function.
(2) Find a harmonic conjugate v of u.

Solution.

– The first partial derivatives of u are:

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2 - 2y \\ \\ \dfrac{\partial u}{\partial y} = -2x \end{cases}$$

Hence $\dfrac{\partial^2 u}{\partial x^2} = 0 = \dfrac{\partial^2 u}{\partial y^2}$. Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, consequently u is a harmonic function.

– If v is a harmonic conjugate of u, then, using CRE, we have

$$\begin{cases} \dfrac{\partial u}{\partial x} = 2 - 2y = \dfrac{\partial v}{\partial y} & (1) \\ \\ \dfrac{\partial u}{\partial y} = -2x = -\dfrac{\partial v}{\partial x} & (2) \end{cases}$$

From Equation (1), we deduce that $v(x, y) = 2y - y^2 + \alpha(x)$. Now, substituting this value in Equation (2), we get $-2x = -\alpha'(x)$. This leads to $\alpha(x) = -x^2 + c$, for some constant $c \in \mathbb{R}$.

Therefore,

$$v(x, y) = 2y - y^2 - x^2 + c,$$

so $v(x, y) = 2y - y^2 - x^2$ is a harmonic conjugate of u. □