SOLUTION – MASTER

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
SOLUTION Math 202 Final Exam
The First Semester of 2011-2012 (111)
Time Allowed: 180 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.
- Write MCQ Answers on the front page.

Written Questions

Question #	Marks	Maximum Marks
1		12
2		20
3		16
4		20
5		16
Total		/84

Multiple Choice Questions

Question #	Student Answer	Marks	Maximum Marks
1			7
2			7
3			7
4			7
5			7
6			7
7			7
8			7
Total			/56
Grand Total			/140

Q:1 (12 pints) Determine singular points of the differential equation

$$x^3(x^2-16)(x-1)^2y'' + 3x(x-1)y' + 5(x+4)y = 0.$$

Classify each singular point as regular or irregular.

Sol: $x^3(x^2-16)(x-1)^2 = 0 \Rightarrow x = 0, 1, -4, 4$ are the singular points.——(4 points)

$$P(x) = \frac{3x(x-1)}{x^3(x^2-16)(x-1)^2} = \frac{3}{x^2(x-4)(x+4)(x-1)}$$

$$Q(x) = \frac{5(x+4)}{x^3(x^2-16)(x-1)^2} = \frac{5}{x^3(x-4)(x-1)^2}$$

Check $x = 0$, $p(x) = xP(x) = \dfrac{3}{x(x-4)(x+4)(x-1)}$, not analytic at $x = 0$

$\Rightarrow$ $x = 0$ is an irregular singular point————————————(2 points).

Check $x = 1$, $p(x) = (x-1)P(x) = \dfrac{3}{x^2(x-4)(x+4)}$,

$$q(x) = (x-1)^2Q(x) = \frac{5}{x^3(x-4)}$$

$\Rightarrow$ $x = 1$ is a regular singular point————————————(2 points).

Check $x = 4$, $p(x) = (x-4)P(x) = \dfrac{3}{x^2(x+4)(x-1)}$,

$$q(x) = (x-4)^2Q(x) = \frac{5(x-4)}{x^3(x-1)^2}$$

$\Rightarrow$ $x = 4$ is a regular singular point————————————(2 point).

Check $x = -4$, $p(x) = (x+4)P(x) = \dfrac{3}{x^2(x-4)(x-1)}$,

$$q(x) = (x+4)^2Q(x) = \frac{5(x+4)^2}{x^3(x-4)(x-1)^2}$$

$\Rightarrow$ $x = -4$ is a regular singular point————————————(2 points).

Q:2 (20 points) Find two linearly independent power series solutions of $y'' + xy' + 3y = 0$ about the ordinary point $x = 0$. Give the first three nonzero terms for each series solution.

Sol: Let $y = \sum\limits_{n=0}^{\infty} c_n x^n$, then $y' = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum\limits_{n=2}^{\infty} n(n-1) c_n x^{n-2}$——(3 points)

Substituting in the equation $y'' + xy' + 3y = 0$, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + 3\sum_{n=0}^{\infty} c_n x^n = 0,$$

$$2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + 3c_0 + 3\sum_{n=1}^{\infty} c_n x^n = 0, \text{——(4 points)}$$

$$(2c_2 + 3c_0) + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} (k+3)c_k x^k = 0, \text{ This gives}$$

$$2c_2 + 3c_0 = 0 \text{ and } c_{k+2} = -\frac{(k+3)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots \text{———————(4 points)}$$

$$c_2 = -\frac{3}{2}c_0, \quad c_3 = -\frac{2}{3}c_1, \quad c_4 = -\frac{5}{4 \times 3}c_2 = \frac{5}{8}c_0, \quad c_5 = -\frac{3}{10}c_3 = \frac{1}{5}c_1 \text{———-(4 points)}$$

The general solution of the differential equation is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots$$

$$= c_0 + c_1 x - \frac{3}{2}c_0 x^2 - \frac{2}{3}c_1 x^3 + \frac{5}{8}c_0 x^4 + \frac{1}{5}c_1 x^5 \ldots$$

$$= c_0\left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 + \ldots\right) + c_1\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \ldots\right) \text{——(3 points)}$$

The two linearly independent solutions are

$$y_1 = 1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 + \ldots \text{ and } y_2 = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \ldots \text{ ——(2 points)}$$

Q:3 (16 points) Solve the differential equation $X' = AX$, where $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Sol: $|A-\lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)^2 = 0 \Rightarrow \lambda = 3, 2, 2$ —(6 points)

For $\lambda = 3 : (A-3I)X = \mathbf{0} \quad \Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = x_2, \; x_3 = 0 \quad \Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ————(2 points)

For $\lambda = 2 : (A-2I)X = \mathbf{0} \quad \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = -x_2, \; x_3 = 0 \quad \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ————(2 points)

Now we solve $(A-2I)P = X_2 \quad \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow \quad p_1 + p_2 - p_3 = 1, \; p_3 = -1 \quad \Rightarrow P = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ ————(3 points)

$X = c_1 X_1 e^{3t} + c_2 X_2 e^{2t} + c_3 \left[X_2 t + P \right] e^{2t}$

$= c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \left(t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right) e^{2t}$ ————(3 points)

Q:4 (20 points) Solve the initial value problem $X' = AX$, $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $A = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$

Sol: $|A - \lambda I| = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2 = 0$

$\Rightarrow \quad \lambda^2 + 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda = -2 \pm i$ ——————————(4 points)

For $\lambda = -2 + i : (A - (-2+i)I)X = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad (-1-i)x_1 - x_2 = 0, \;\; 2x_1 + (1-i)x_2 = 0$ ——————————(4 points)

$\Rightarrow \quad X_1 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$, there can be other choices. —-(4 points)

Let $B_1 = Re(X_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $B_2 = Im(X_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, then

$$X = c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t \right) e^{-2t} + c_2 \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sin t \right) e^{-2t}$$

$$= c_1 \begin{bmatrix} \cos t \\ -\cos t + \sin t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sin t \\ -\cos t - \sin t \end{bmatrix} e^{-2t}$$ ——————————(4 points)

Now $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad c_1 = 1, \; c_2 = -2$

$$X = \begin{bmatrix} \cos t \\ -\cos t + \sin t \end{bmatrix} e^{-2t} - 2 \begin{bmatrix} \sin t \\ -\cos t - \sin t \end{bmatrix} e^{-2t} = \begin{bmatrix} \cos t - 2\sin t \\ \cos t + 3\sin t \end{bmatrix} e^{-2t}$$ -(4 points)

Q:5 (16 points) Find X_p for the system $X' = AX + F(t)$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, $F(t) = \begin{bmatrix} 70e^{4t} \\ 20e^{t} \end{bmatrix}$ and $X_c = c_1 \begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$.

Sol: From X_c, we get the fundamental matrix $\Phi = \begin{bmatrix} e^{-3t} & 2e^{2t} \\ -2e^{-3t} & e^{2t} \end{bmatrix}$ —————(4 points)

$$|\Phi| = \begin{vmatrix} e^{-3t} & 2e^{2t} \\ -2e^{-3t} & e^{2t} \end{vmatrix} = 5e^{-t},\ \Phi^{-1} = \tfrac{1}{5}e^{t} \begin{bmatrix} e^{2t} & -2e^{2t} \\ 2e^{-3t} & e^{-3t} \end{bmatrix} = \tfrac{1}{5} \begin{bmatrix} e^{3t} & -2e^{3t} \\ 2e^{-2t} & e^{-2t} \end{bmatrix} \text{(4points)}$$

$$\Phi^{-1}F(t) = \tfrac{1}{5} \begin{bmatrix} e^{3t} & -2e^{3t} \\ 2e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 70e^{4t} \\ 20e^{t} \end{bmatrix} = \begin{bmatrix} 14e^{7t} - 8e^{4t} \\ 28e^{2t} + 4e^{-t} \end{bmatrix} \text{———(3 points)}$$

$$\int \Phi^{-1}F(t)dt = \begin{bmatrix} 2e^{7t} - 2e^{4t} \\ 14e^{2t} - 4e^{-t} \end{bmatrix} \text{———(2 points)}$$

$$X_p = \Phi \int \Phi^{-1}F(t)dt = \begin{bmatrix} e^{-3t} & 2e^{2t} \\ -2e^{-3t} & e^{2t} \end{bmatrix} \begin{bmatrix} 2e^{7t} - 2e^{4t} \\ 14e^{2t} - 4e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{4t} - 2e^{t} + 28e^{4t} - 8e^{t} \\ -4e^{4t} + 4e^{t} + 14e^{4t} - 4e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 30e^{4t} - 10e^{t} \\ 10e^{4t} \end{bmatrix} = 10 \begin{bmatrix} 3e^{4t} - e^{t} \\ e^{4t} \end{bmatrix} \text{———(3 points)}$$

1. Integrating factor that makes the differential equation $(-xy\sin x + 2y\cos x)dx + 2x\cos x dy = 0$ EXACT is:

(a) $(\sec x)^{\frac{1}{2}}$

(b) $(\tan x)^{\frac{1}{2}}$

(c) $\frac{1}{2}\ln(\sec x)$

(d) $\frac{1}{2}\sec x$

(e) $\frac{1}{2}\tan x$

2. If $y(x)$ is a solution of $(x + ye^{\frac{y}{x}})dx - xe^{\frac{y}{x}}dy = 0$ with $y(1) = 0$, then $y(e^2)$ is equal to:

(a) $e^2 \ln 3$

(b) $e \ln 3$

(c) $3 \ln 3$

(d) $e \ln 9$

(e) $e^2 \ln 9$

3. After 6 hours, a radioactive material has decreased by 87.5% (remained 12.5%). What is the half life of the material?

(a) 2 hours

(b) 4 hours

(c) 6 hours

(d) 8 hours

(e) ∞ hours

4. If $y(x)$ is the solution of the initial value problem $3y''' + 2y'' = 0$, $y(0) = -1,\ y'(0) = 0,\ y''(0) = 1$, then $y(\frac{3}{2})$ is equal to:

(a) $\dfrac{9}{4e} - 1$

(b) $\dfrac{3}{4e} - 1$

(c) $\dfrac{3}{4e} + 1$

(d) $\dfrac{9}{4e} + 1$

(e) $\dfrac{3}{2e} - 1$

5. Which one of the following functions is annihilated by the operator $(D+1)(D^2-6D+25)$

(a) $e^{-x}+e^{3x}\cos 4x$

(b) $e^{-x}+xe^{3x}\cos 4x$

(c) $e^{-x}+e^{5x}+xe^{5x}$

(d) $e^{-x}+e^{-5x}+xe^{-5x}$

(e) $e^{-x}+e^{-3x}\cos 5x+e^{-3x}\sin 5x$

6. If $y_c = c_1\cos 2x + c_2\sin 2x$ is complementary function of the equation $4y''+16y=\csc 2x$, then a particular solution is given by:

(a) $-\frac{1}{8}x\cos 2x+\frac{1}{16}\sin 2x\ln|\sin 2x|$

(b) $-\frac{1}{16}x\cos 2x+\frac{1}{8}\sin 2x\ln|\sin 2x|$

(c) $-\frac{1}{8}x\cos 2x-\frac{1}{8}\sin 2x\ln|\sec 2x|$

(d) $-\frac{1}{8}x\cos 2x+\frac{1}{16}\sin 2x\ln|\cos 2x|$

(e) $-\frac{1}{16}x\cos 2x+\frac{1}{8}\sin 2x\ln|\csc 2x|$

7. If we convert the Cauchy-Euler equation $-2x^2y'' + xy' - 2y = 0$ into an equation with constant coefficients $y'' + ay' + by = 0$, then $a + b$ is equal to:

(a) $-\frac{1}{2}$

(b) $-\frac{3}{2}$

(c) $\frac{1}{2}$

(d) $-\frac{1}{3}$

(e) $-\frac{2}{3}$

8. If $y_{p_1} = xe^x$ is a particular solution of $y'' - y' = e^x$
and $y_{p_2} = \frac{1}{2}(\cos x - \sin x)$ is a particular solution of $y'' - y' = \sin x$,
then $y_p = \sin x - \cos x - xe^x$ is a particular solution of:

(a) $y'' - y' = -2\sin x - e^x$

(b) $y'' - y' = \sin x - \cos x + e^x$

(c) $y'' - y' = 2\sin x - e^x$

(d) $y'' - y' = e^x - \cos x$

(e) $y'' - y' = 2e^x - 2\sin x$