

Part I [76 pts]

(Written: Provide all necessary steps required in the solution.)

- Q1. Find the extreme values of $f(x, y) = xy^2$ on the ellipse $2x^2 + y^2 = 6$ by using the Lagrange multipliers. (10 pts)

Solution: Let $g(x, y) = 2x^2 + y^2$.

• We solve, for x, y and λ , the following system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = \end{cases} \Rightarrow \begin{cases} y^2 = \lambda(4x) & \textcircled{1} \\ 2xy = \lambda(2y) \rightarrow 2y(x - \lambda) = 0 & \textcircled{2} \\ 2x^2 + y^2 = 6 & \textcircled{3} \end{cases} \textcircled{3}$$

From $\textcircled{2}$, $y = 0$ or $\lambda = x$

• If $y = 0 \Rightarrow$ in $\textcircled{3}$, we find $x = \pm\sqrt{3} \Rightarrow (\pm\sqrt{3}, 0) \textcircled{1}$

• If $\lambda = x \Rightarrow$ in $\textcircled{1}$, we find $y^2 = 4x^2 \Rightarrow$ in $\textcircled{3}$, we find

$$2x^2 + 4x^2 = 6 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\Rightarrow (1, \pm 2) \textcircled{1} \text{ and } (-1, \pm 2) \textcircled{1}$$

• Now, we compare the values of f at these points:

$$f(\pm\sqrt{3}, 0) = 0$$

$$f(1, \pm 2) = 4$$

$$f(-1, \pm 2) = -4$$

$\Rightarrow f$ has max. value 4 at the points $(1, \pm 2) \textcircled{2}$

and f has min. value -4 at the points $(-1, \pm 2) \textcircled{2}$

Q2. (i) Find the Cartesian Equation of the parametric curve $x = 2 \cos 2t$, $y = 3 \sin 2t$; $-\pi/4 \leq t \leq \pi/4$. (4 pts)

Solution:

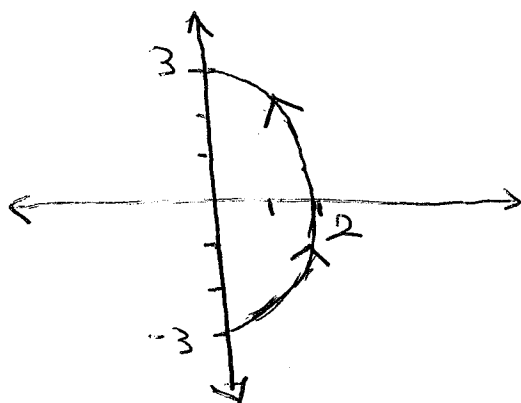
(a) • $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = (\cos 2t)^2 + (\sin 2t)^2 = 1$

$\Rightarrow \boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$ _____ (3)

where $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4} \Rightarrow 0 \leq \cos 2t \leq 1$
 $\Rightarrow \boxed{0 \leq x \leq 2}$ _____ (1)

and $-1 \leq \sin 2t \leq 1 \Rightarrow \boxed{-3 \leq y \leq 3}$

(ii) Sketch the above curve in the xy -plane. Indicate with an arrow the direction in which the curve is traced as the parameter t increases. (4 pts)



- (2) - shape
- (1) - direction
- (1) - intercepts
-

(iii) Set up the integral that represents the surface area of the solid obtained by revolving the above curve about the y -axis. (4 pts)

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad \text{_____ (2)}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\pi \cos 2t \sqrt{(-4\sin 2t)^2 + (6\cos 2t)^2} dt$$

$$= 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2t \sqrt{16 + 20\cos^2 2t} dt \quad \text{_____ (2)}$$

Q3. If $f(x, y, z) = \ln(zx^2y^2)$, $x = \cos t$, $y = \ln(t^2 + \theta)$, $z = \theta^2$, find $\partial f / \partial t$ at $(t, \theta) = (1, \pi)$. (8 pts)

Solution:

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} \\ &= \left(\frac{2}{x}\right)(-\theta \sin t \theta) + \left(\frac{2}{y}\right)\left(\frac{2t}{t^2 + \theta}\right) + \left(\frac{1}{z}\right)(0) \\ \Rightarrow \frac{\partial f}{\partial t} \Big|_{\substack{t=1 \\ \theta=\pi}} &= \left(\frac{2}{\cos \pi}\right)(-\pi \sin \pi) + \left(\frac{2}{\ln(1+\pi)}\right)\left(\frac{2}{1+\pi}\right) = \frac{4}{(1+\pi) \ln(1+\pi)} \end{aligned}$$

Q4. Find the equation of the plane that contains the points $(1, 2, -3)$, $(3, 1, 0)$, and $(2, 3, -7)$.

(8 pts)

Solution:

$$\begin{aligned} \bullet \vec{AB} &= \langle 2, -1, 3 \rangle \quad \text{and} \quad \vec{AC} = \langle 1, 1, -4 \rangle \\ \Rightarrow \vec{n} &= \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 1 & -4 \end{vmatrix} = \hat{i} + 11\hat{j} + 3\hat{k} \end{aligned}$$

Then, \vec{n} is normal to the plane \Rightarrow the equation of the plane is:

$$1(x-1) + 11(y-2) + 3(z+3) = 0$$

$$\Rightarrow \boxed{x + 11y + 3z - 14 = 0} \quad \text{--- (3)}$$

Q5. Consider the function $f(x, y) = x^2 + y^2 + x^2y + 4$.

(i) Show that $(0, 0)$, $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$ are the critical points of $f(x, y)$.

(4 pts)

Solution

• We solve the system $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2xy = 0 \\ 2y + x^2 = 0 \end{cases} \begin{matrix} \textcircled{1} \\ \textcircled{2} \end{matrix} \quad \textcircled{1}$

From $\textcircled{2}$, $y = -\frac{1}{2}x^2 \Rightarrow$ in $\textcircled{1}$, we have:

$$2x + 2x\left(-\frac{1}{2}x^2\right) = 0 \Rightarrow 2x - x^3 = 0 \Rightarrow x(2 - x^2) = 0$$

$$\Rightarrow x = 0, \sqrt{2}, -\sqrt{2} \Rightarrow y = 0, -1, -1 \text{ respectively}$$

\Rightarrow The critical points are $(0, 0)$, $(\sqrt{2}, -1)$, $(-\sqrt{2}, -1)$.

(ii) Find the local minimum and maximum value(s) and the saddle point(s) of $f(x, y)$, if any.

(6 pts)

• $f_{xx} = 2 + 2y$ / $f_{yy} = 2$ / $f_{xy} = 2x$ } $\textcircled{3}$

$$\Rightarrow D = f_{xx}f_{yy} - (f_{xy})^2 = 4 + 4y - 4x^2$$

• $D(0, 0) = 4 > 0$ and $f_{xx}(0, 0) = 2 > 0$

$\Rightarrow f(0, 0) = 4$ is a local minimum value. $\textcircled{1}$

• $D(\sqrt{2}, -1) = -8 < 0 \Rightarrow (\sqrt{2}, -1)$ is a saddle point. $\textcircled{1}$

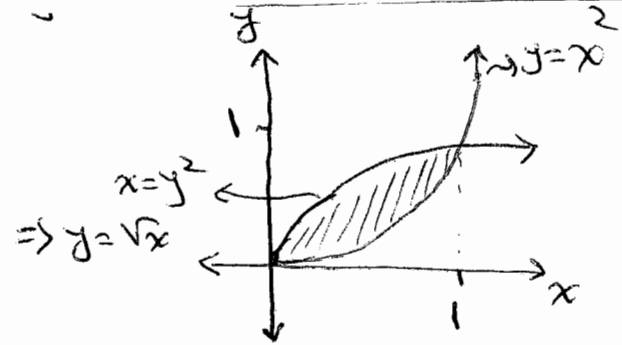
• $D(-\sqrt{2}, -1) = -8 < 0 \Rightarrow (-\sqrt{2}, -1)$ is a saddle point. $\textcircled{1}$

Q6. Find the value of the integral $\iiint_E xy dV$ where E is bounded by the parabolic cylinders $x = y^2$ and $y = x^2$, and the planes $z = 0$ and $z = x + y$. (10pts)

Solution:

$$\iiint_E xy dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy dz dy dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [x^2 y + x y^2] dy dx$$



$$= \int_0^1 \left[\frac{1}{2} x^2 y^2 + \frac{1}{3} x y^3 \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[\frac{1}{2} x^3 + \frac{1}{3} x^{\frac{5}{2}} - \frac{1}{2} x^6 - \frac{1}{3} x^7 \right] dx$$

$$= \left[\frac{1}{8} x^4 + \frac{2}{21} x^{\frac{7}{2}} - \frac{1}{14} x^7 - \frac{1}{24} x^8 \right]_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24} = \frac{3}{28}$$

- Q7. A solid lies within both the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = 4 - x^2 - y^2$, and above the xy -plane. Only **Set up** the integral in **cylindrical coordinates** that represents the volume of the solid. (8 pts)

Using $x^2 + y^2 = r^2$, this solid is described in cylindrical coordinates by: $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq z \leq 4 - r^2\}$

$\Rightarrow V = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_0^{4-r^2} r \, dz \, dr \, d\theta$

Q8. Evaluate the integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} z \sqrt{x^2+y^2+z^2} dz dy dx$.

(10 pts)

Solution:

using $x = \rho \sin \phi \cos \theta$ / $y = \rho \sin \phi \sin \theta$ / $z = \rho \cos \phi$

and $x^2 + y^2 + z^2 = \rho^2$, we find that

$$E = \left\{ (x, y, z) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, 0 \leq z \leq \sqrt{1-x^2-y^2} \right\}$$

is the unit hemisphere above the xy -plane.

$$\Rightarrow E = \left\{ (\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \rho \leq 1 \right\}$$

$$\Rightarrow \iiint_E z \sqrt{x^2+y^2+z^2} dz dy dx = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \cos \phi) (\rho) \rho^2 \sin \phi d\rho d\phi d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^4 \sin 2\phi d\rho d\phi d\theta$$

$$= \frac{1}{10} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin 2\phi d\phi d\theta = \frac{1}{10} \int_0^{2\pi} d\theta = \boxed{\frac{\pi}{5}}$$