Part I [76 pts] (Written: Provide all necessary steps required in the solution.)

Q1. Find the **extreme values** of $f(x,y) = xy^2$ on the **ellipse** $2x^2 + y^2 = 6$ by using the **Lagrange multipliers.** (10 pts)

Solution:- Let $g(x,y) = 2x^2+y^2$.

- We solve, for x, y and λ, the following system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = \end{cases} \Rightarrow \left.\begin{cases} y^2 = \lambda(4x) & (1) \\ 2xy = \lambda(2y) \longrightarrow 2y(x-\lambda) = 0 & (2) \\ 2x^2+y^2 = 6 & (3) \end{cases}\right\} - (3)$$

From (2), $y = 0$ or $\lambda = x$

- If $y = 0 \Rightarrow$ in (3), we find $x = \pm\sqrt{3} \Rightarrow \boxed{(\pm\sqrt{3}, 0)}$ — (1)
- If $\lambda = x \Rightarrow$ in (1), we find $y^2 = 4x^2 \Rightarrow$ in (3), we find

$$2x^2 + 4x^2 = 6 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\Rightarrow \boxed{(1, \pm 2)} \text{ (1) and } \boxed{(-1, \pm 2)} \text{ (1)}$$

- Now, we compare the values of f at these points:-

$$f(\pm\sqrt{3}, 0) = 0$$

$$f(1, \pm 2) = 4$$

$$f(-1, \pm 2) = -4$$

$\Rightarrow f$ has max. value 4 at the points $(1, \pm 2)$ — (2)

and f has min. value -4 at the points $(-1, \pm 2)$ — (2)

Q2. (i) Find the **Cartesian Equation** of the **parametric curve** $x = 2\cos 2t,\ y = 3\sin 2t;\ -\pi/4 \le t \le \pi/4$.

(4 pts)

Solution:

(a) • $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = (\cos 2t)^2 + (\sin 2t)^2 = 1$

$\Rightarrow \boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$ ——— (3)

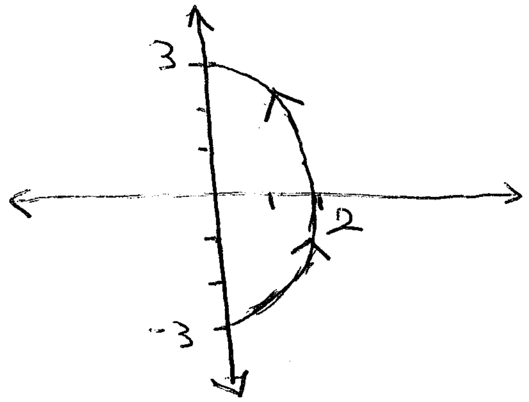
where $-\frac{\pi}{4} \le t \le \frac{\pi}{4} \Rightarrow 0 \le \cos 2t \le 1$

$\Rightarrow \boxed{0 \le x \le 2}$ — (1)

and $-1 \le \sin 2t \le 1 \Rightarrow \boxed{-3 \le y \le 3}$

(ii) Sketch the above curve in the *xy*-plane. **Indicate** with an arrow the direction in which the curve is traced as the parameter *t* increases.

(4 pts)



(2) – shape

(1) – direction

(1) – intercepts

(iii) Set up the integral that represents the **surface area** of the solid obtained by revolving the above curve **about the *y*-axis**.

(4 pts)

$$S = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{———} \ (2)$$

$$= \int_{-\pi/4}^{\pi/4} 4\pi \cos 2t \sqrt{(-4\sin 2t)^2 + (6\cos 2t)^2}\, dt$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} \cos 2t \sqrt{16 + 20\cos^2 2t}\, dt \quad \text{———} \ (2)$$

Q3. If $f(x,y,z)=\ln(zx^2y^2), x=\cos t\theta, y=\ln(t^2+\theta), z=\theta^2$, find $\partial f/\partial t$ at $(t,\theta)=(1,\pi)$. (8 pts)

Solution:-

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$=\left(\frac{2}{x}\right)(-\theta\sin t\theta)+\left(\frac{2}{y}\right)\left(\frac{2t}{t^2+\theta}\right)+\cancel{\left(\frac{1}{z}\right)(0)}$$

2 2 2

$$\Rightarrow \left.\frac{\partial f}{\partial t}\right|_{\substack{t=1\\ \theta=\pi}}=\left(\frac{2}{\cos\pi}\right)(-\pi\sin\pi)+\left(\frac{2}{\ln(1+\pi)}\right)\left(\frac{2}{1+\pi}\right)=\boxed{\frac{4}{(1+\pi)\ln(1+\pi)}}$$

2

6

Q4. Find the **equation of the plane** that contains the points $\underbrace{(1,2,-3)}_{A},\underbrace{(3,1,0)}_{B}$, and $\underbrace{(2,3,-7)}_{C}$. (8 pts)

Solution:-

1 1

- $\vec{AB}=\langle 2,-1,3\rangle$ and $\vec{AC}=\langle 1,1,-4\rangle$

$$\Rightarrow \vec{n}=\vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-1&3\\1&1&-4\end{vmatrix}=\hat{i}+11\hat{j}+3\hat{k}$$

3

Then, $\vec{n}$ is normal to the plane $\Rightarrow$ the equation of the plane is:-

$$1(x-1)+11(y-2)+3(z+3)=0$$

$$\Rightarrow \boxed{x+11y+3z-14=0}$$ 3

Q5. Consider the function $f(x,y) = x^2 + y^2 + x^2y + 4$.

(i) Show that (0,0), $(\sqrt{2},-1)$ and $(-\sqrt{2},-1)$ are the **critical points** of $f(x,y)$. (4 pts)

Solution:

• We solve the system $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \left.\begin{cases} 2x + 2xy = 0 & (1) \\ 2y + x^2 = 0 & (2) \end{cases}\right\}$ — 1

from (2), $y = -\frac{1}{2}x^2 \Rightarrow$ in (1), we have:

$2x + 2x(-\frac{1}{2}x^2) = 0 \Rightarrow 2x - x^3 = 0 \Rightarrow x(2-x^2) = 0$

$\Rightarrow x = 0, \sqrt{2}, -\sqrt{2} \Rightarrow y = 0, -1, -1$ respectively

1 1 1

$\Rightarrow$ The critical points are $(0,0), (\sqrt{2},-1), (-\sqrt{2},-1)$.

(ii) Find the **local minimum and maximum value(s)** and the **saddle point(s)** of $f(x,y)$, if any. (6 pts)

• $f_{xx} = 2+2y$ / $f_{yy} = 2$ / $f_{xy} = 2x$

$\Rightarrow D = f_{xx}f_{yy} - (f_{xy})^2 = 4+4y-4x^2$ — 3

• $D(0,0) = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$

$\Rightarrow f(0,0) = 4$ is a local minimum value. — 1

• $D(\sqrt{2},-1) = -8 < 0 \Rightarrow (\sqrt{2},-1)$ is a saddle point. 1

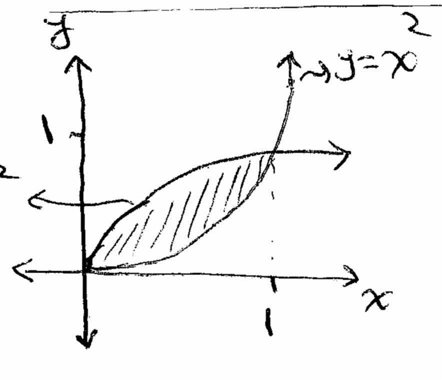
• $D(-\sqrt{2},-1) = -8 < 0 \Rightarrow (-\sqrt{2},-1)$ is a saddle point. 1

Q6. Find the **value of the integral** $\iiint_E xy\,dV$ where E is **bounded by** the parabolic cylinders $x = y^2$ and $y = x^2$, and the planes $z = 0$ and $z = x + y$. (10pts)

Solution

$$\iiint_E xy\,dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\,dz\,dy\,dx$$

(1) (1) (1) (1) (1) (1)



$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [x^2y + xy^2]\,dy\,dx \quad (1)$$

$$= \int_0^1 \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{3}xy^3\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[\tfrac{1}{2}x^3 + \tfrac{1}{3}x^{\frac{5}{2}} - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^7\right] dx \quad (1)$$

$$= \left.\tfrac{1}{8}x^4 + \tfrac{2}{21}x^{\frac{7}{2}} - \tfrac{1}{14}x^7 - \tfrac{1}{24}x^8\right]_0^1 = \tfrac{1}{8} + \tfrac{2}{21} - \tfrac{1}{14} - \tfrac{1}{24} = \boxed{\tfrac{3}{28}}$$

(1)

[redacted] 6 (1)

Q7. A solid lies within both the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = 4 - x^2 - y^2$, and above the *xy*-plane. Only **Set up** the integral in **cylindrical coordinates** that represents the volume of the solid.

(8 pts)

Using $x^2+y^2=r^2$, this solid is described in cylindrical coordinates by :- $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 0 \le z \le 4-r^2\}$

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$$\Rightarrow V = \iiint_E dV = \int_0^{2\pi}\int_0^{1}\int_0^{4-r^2} r\,dz\,dr\,d\theta$$

1 1 1 2

Q8. Evaluate the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dzdydx$. (10 pts)

Solution:

using $x=\rho\sin\phi\cos\theta$ / $y=\rho\sin\phi\sin\theta$ / $z=\rho\cos\phi$

and $x^2+y^2+z^2=\rho^2$, we find that

$$E=\{(x,y,z) \mid -1\le x\le 1,\ -\sqrt{1-x^2}\le y\le\sqrt{1-x^2},\ 0\le z\le\sqrt{1-x^2-y^2}\}$$

is the unit hemisphere above the xy-plane.

$$\Rightarrow E=\{(\rho,\theta,\phi) \mid 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2},\ 0\le\rho\le 1\}$$

$$\Rightarrow \iiint_E z\sqrt{x^2+y^2+z^2}\,dzdydx=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \underline{(\rho\cos\phi)(\rho)\,\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^4\sin 2\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{1}{10}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin 2\phi\,d\phi\,d\theta=\frac{1}{10}\int_0^{2\pi}d\theta=\boxed{\frac{\pi}{5}}$$