MATH 201. Solutions Exam II

Part I [52 pts] **(Written: Provide all necessary steps required in the solution.)**

Q1. **(i)** Find an equation of the plane passing through the points (1,2,3), (− 1 ,2,0) and perpendicular to the plane $x + 2y + 3z = 1$. (7 pts)

- A vector normal to the plane $x+2y+3z=1$ is

1 $$\vec{n}_1 = \langle 1, 2, 3\rangle$$

- A vector passing through the points (1,2,3), (−1,2,0) is

1 $$\vec{v} = \langle 2, 0, 3\rangle$$

- A vector $\vec{n}$ normal to the required plane can be found by:

3 $$\vec{n} = \vec{n}_1 \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} = 6\hat{i} + 3\hat{j} - 4\hat{k}$$

Using the point (1,2,3) and the normal $\vec{n} = \langle 6, 3, -4\rangle$, we find the equation of the required plane to be

2 $$6(x-1) + 3(y-2) - 4(z-3) = 0$$

$$\Rightarrow \boxed{6x + 3y - 4z = 0}$$

(ii) Find the **distance** between the **planes**: $x - 2y + 3z = 1$ and $-2x + 4y - 6z = 1$. (5 pts)

- We choose the point (1,0,0) on the first plane.
- We write the equation of the 2nd plane in the general form as

1 $$-2x + 4y - 6z - 1 = 0$$

- We use the formula to find the distance

$$D = \frac{|-2(1) + 4(0) - 6(0) - 1|}{\sqrt{4 + 16 + 36}} = \boxed{\frac{3}{\sqrt{56}}}$$

2 1

Q2. Let $f(x, y) = \ln\sqrt{16 - 4x^2 - y^2}$

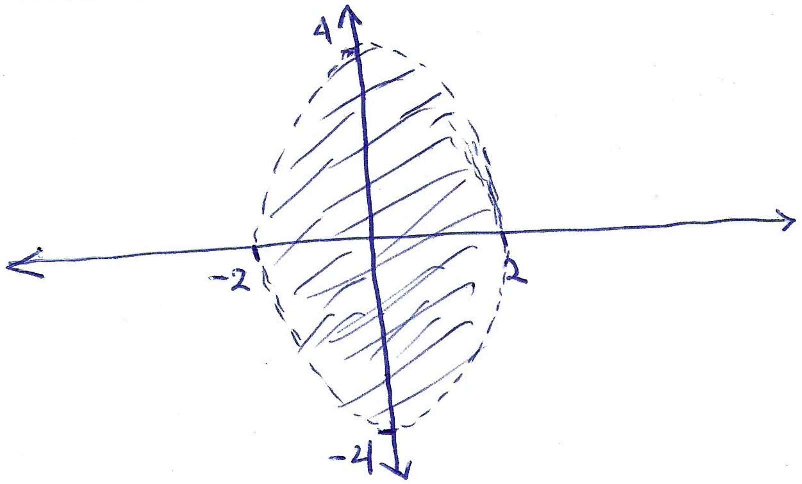
(i) Find and sketch the domain of f. (3+2 pts)

2

$D = \{(x,y) : 16 - 4x^2 - y^2 > 0\}$

$= \{(x,y) : 4x^2 + y^2 < 16\} = \{(x,y) : \frac{x^2}{4} + \frac{y^2}{16} < 1\}$ 1

This is the interior of the ellipse sketched below



2

(ii) Find the range of f. (2 pts)

1 $0 < 16 - 4x^2 - y^2 \leq 16 \Rightarrow 0 < \sqrt{16 - 4x^2 - y^2} \leq 4$

$\Rightarrow$ $f(x) = \ln\sqrt{16 - 4x^2 - y^2}$ has the range

1 $R = (-\infty, \ln 4]$

(iii) Write an equation of the level curve of f which passes through the point (1,1). (3 pts)

1 $f(1,1) = \ln\sqrt{11}$

$\Rightarrow$ The level curve that passes through (1,1) has an equation

2 $\ln\sqrt{16 - 4x^2 - y^2} = \ln\sqrt{11}$

$\Rightarrow$ $16 - 4x^2 - y^2 = 11 \Rightarrow 4x^2 + y^2 = 5$.

Q3. Find **parametric equations** of the **normal line** to the **surface** $\ln\left(\frac{x}{y-z}\right) = x - 1$ at the point (1,4,3).

(8 pts)

Let $F(x,y,z) = \ln\left(\frac{x}{y-z}\right) - x + 1$ 1

$\Rightarrow$ A normal vector to the given surface at the given point is

$\vec{n} = \nabla F(1,4,3) = \left\langle \frac{1}{x} - 1, \frac{1}{z-y}, \frac{1}{y-z} \right\rangle \Big|_{(1,4,3)}$

1 2

$= \langle 0, -1, 1 \rangle$ 1

$\Rightarrow$ The parametric equations of the normal line at (1,4,3) are

$\boxed{x = 1}$ / $\boxed{y = 4 - t}$ / $\boxed{z = 3 + t}$

1 1 1

Q4. The values of $z = f(x, y)$ and its partial derivatives at $(2, -2)$ are given in the following table:

$f(2,-2)$	$f_x(2,-2)$	$f_y(2,-2)$	$f_{xx}(2,-2)$	$f_{yy}(2,-2)$	$f_{xy}(2,-2)$
2	-5	3	4	7	-3

If $x = r^2 + s^2$ and $y = 2rs$, then find

(i) $\left.\dfrac{\partial z}{\partial s}\right]_{(r,s)=(1,-1)}$ (4 pts)

- $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2s\frac{\partial z}{\partial x} + 2r\frac{\partial z}{\partial y}$$

- $\boxed{\text{When } r=1, s=-1 \Rightarrow x=2, y=-2}$

$$\Rightarrow \left.\frac{\partial z}{\partial s}\right|_{(r,s)=(1,-1)} = 2(-1)(-5) + 2(1)(3) = \boxed{16}$$

(ii) $\left.\dfrac{\partial^2 z}{\partial r \partial s}\right]_{(r,s)=(1,-1)}$ (6 pts)

- $$\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial s}\right] = \frac{\partial}{\partial r}\left[2s\frac{\partial z}{\partial x} + 2r\frac{\partial z}{\partial y}\right]$$

$$= 2s\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\right] + 2\frac{\partial z}{\partial y} + 2r\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]$$

$$= 2s\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right] + 2\frac{\partial z}{\partial y} + 2r\left[\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right]$$

$$= 2s\left[2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right] + 2\frac{\partial z}{\partial y} + 2r\left[2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right]$$

$$= 4sr\frac{\partial^2 z}{\partial x^2} + (4s^2 + 4r^2)\frac{\partial^2 z}{\partial x \partial y} + 4sr\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial y}$$

- When $r=1, s=-1 \Rightarrow x=2, y=-2 \Rightarrow$

$$\left.\frac{\partial^2 z}{\partial r \partial s}\right|_{(r,s)=(1,-1)} = 4(-1)(1)(4) + (4(-1)^2 + 4(1)^2)(-3) + 4(-1)(1)(7) + 2(3)$$

$$= -16 - 24 - 28 + 6 = \boxed{-62}$$

Q5. Find the **absolute maximum** and **absolute minimum** of $f(x,y) = x(y^2-1)$ on the **region** $D = \{(x,y) : x^2+y^2 \le 28\}$. (12 pts)

• We find the critical points inside D:

Deduct 1 pt. if $(0,0)$ is claimed as critical pt

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} y^2-1=0 \\ 2xy=0 \end{cases} \Rightarrow \begin{cases} y=\pm 1 \\ x=0 \text{ or } y=0 \end{cases}$$

2

$\Rightarrow$ The critical points are $\boxed{(0,1) \text{ and } (0,-1)}$ with

$$\boxed{f(0,\pm 1) = 0}$$ 2

• On the boundary $x^2+y^2=28 \Rightarrow f$ becomes

1 $f = x(28-x^2-1) = 27x - x^3$ $\quad -\sqrt{28} \le x \le \sqrt{28}$

1 $\Rightarrow f' = 27-3x^2 = 0 \Rightarrow x = \pm 3 \Rightarrow y = \pm\sqrt{19}$

$\Rightarrow \boxed{f(3,\pm\sqrt{19}) = 54}$ / $\boxed{f(-3,\pm\sqrt{19}) = -54}$ 1

and at the endpoints of the interval, we find

$x = \pm\sqrt{28} \Rightarrow y = 0 \Rightarrow \boxed{f(\sqrt{28},0) = -\sqrt{28}}$ 1

$\boxed{f(-\sqrt{28},0) = \sqrt{28}}$ 1

• Comparing all the above values obtained, we find that

1 f has max. value 54 at $(3,\sqrt{19})$ and $(3,-\sqrt{19})$

1 and f has min. value -54 at $(-3,\sqrt{19})$ and $(-3,-\sqrt{19})$.

KEY-Exam II-4 Versions MATH 201

MCQ #	V-1	MCQ #	V-2	MCQ #	V-3	MCQ #	V-4
1	b	1	e	1	e	1	d
2	d	2	c	2	d	2	b
3	d	3	b	3	e	3	c
4	c	4	d	4	b	4	a
5	e	5	a	5	c	5	c
6	a	6	c	6	c	6	b
7	e	7	a	7	a	7	e
8	b	8	d	8	b	8	e