Math201-111 Exam 1
Solution

Part I [52 pts] (Written: Provide all necessary steps required in the solution.)

Q1. (i) Find d^2y/dx^2 for the parametric curve **C:** $x = 3t^2 - t, \quad y = 2t + t^3$. (5+5 pts)

Sol: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2+2}{6t-1}$ and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{(6t)(6t-1)-(3t^2+2)6}{(6t-1)^3} = \frac{36t^2-6t-18t^2-12}{(6t-1)^3} = \frac{18t^2-6t-12}{(6t-1)^3}$$

(ii) Find the interval (s) where **C** is **concave up**.

Sol: C is concave up if

$$\frac{d^2y}{dx^2} = \frac{18t^2-6t-12}{(6t-1)^3} = \frac{6(3t+2)(t-1)}{(6t-1)^2} > 0$$

t	-2/3	1/6	1	
3t+2	−	+	+	+
6t – 1	−	−	+	+
t – 1	−	−	−	+
$\frac{d^2y}{dx^2}$	−	+	−	+

The curve is concave up for $x \in \left(-\frac{2}{3}, \frac{1}{6}\right) \cup (1, \infty)$

Q2. Consider the vectors $\vec{u} = -3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + 2\vec{j} - 3\vec{k}$ (10 pts)

(i) Find the angle between $\vec{u}$ and $\vec{v}$.

Sol: $cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-3+2-6}{\sqrt{14}\sqrt{14}} = \frac{-7}{14} = -\frac{1}{2} \quad \Longrightarrow \theta = \frac{2\pi}{3}$.

(ii) Find the projection of $\vec{u}$ onto $\vec{v}$

Sol: $proj_{\vec{v}}^{\vec{u}} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{-7}{14}\langle 1,2,-3\rangle = \langle -\frac{1}{2}, -1, \frac{3}{2}\rangle$.

Q3. Use the **scalar triple product** to determine whether the four points: (10 pts)

$$A(1,3,2),\ B(3,-1,6),\ C(5,2,0),\ D(3,6,-4)$$

lie in the **same plane**.

Sol: $\overrightarrow{AB} = \langle 2,-4,4\rangle,\ \overrightarrow{AC} = \langle 4,-1,-2\rangle,\ and\ \overrightarrow{AD} = \langle 2,3,-6\rangle$

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2(6+6) + 4(-24+4) + 4(12+2) = 24 - 80 + 56 = 0.$$

Q4. Find the exact **length** of the **polar curve**: $r = \theta^2, \quad 0 \le \theta \le \pi/4$. (10 pts)

Sol: $s = \int_0^{\frac{\pi}{4}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\frac{\pi}{4}} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{\frac{\pi}{4}} \theta\sqrt{\theta^2 + 4}\, d\theta = \frac{1}{3}(\theta^2+4)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{4}}$

$$= \frac{1}{3}\left(\frac{\pi^2}{16} + 4\right)^{\frac{3}{2}} - \frac{1}{3}4^{\frac{3}{2}} = \frac{1}{3}\left(\frac{\pi^2}{16} + 4\right)^{\frac{3}{2}} - \frac{8}{3}.$$

5. Consider the polar curve **C**: $r = 2 + 4\sin\theta$ (3 + 2 + 2 +5 pts)

(a) Show that **C** is **symmetric** about the vertical line $\theta = \dfrac{\pi}{2}$.

Sol:

$$r(\pi - \theta) = 2 + 4\sin(\pi - \theta) = 2 + 4(\sin\pi\cos\theta - \cos\pi\sin\theta) = 2 + 4\sin\theta = r(\theta)$$

(b) Find the **polar coordinates** of the points where C **intersects** the **polar axis**.

Sol: C intersect polar axis at $\theta = 0, r = 2$, that is (2,0).

C also intersect polar axis at pole, that is r=0.

$$r = 2 + 4\sin\theta = 0 \implies \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

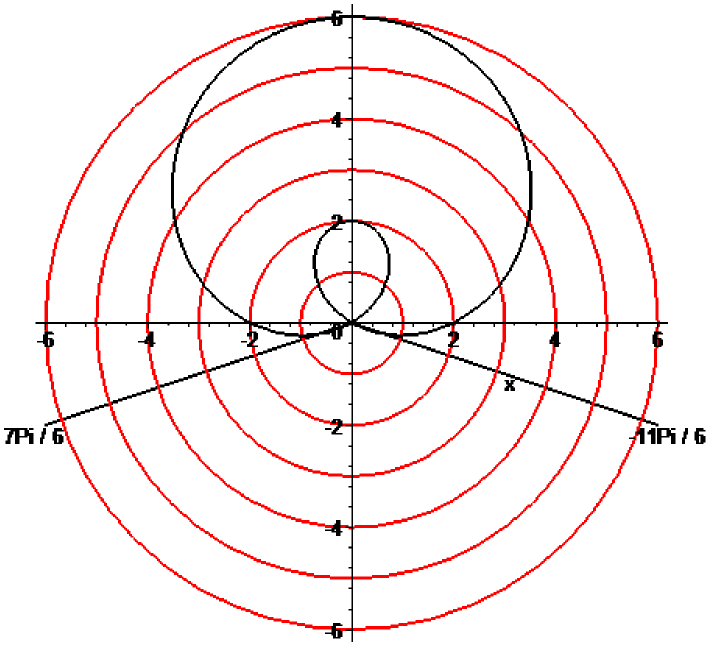
$$\left(0, \frac{7\pi}{6}\right), \left(0, \frac{11\pi}{6}\right).$$

(c) Find the **polar coordinates** of the points where C **intersects** the lines $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{4}$

At $\theta = \frac{\pi}{2}, r = 6$ and at $\theta = \frac{\pi}{4}, r = 2 + 2\sqrt{2}$

The points are $\left(6, \frac{\pi}{2}\right)$ and $\left(2 + 2\sqrt{2}, \frac{\pi}{4}\right)$.

(d) **Plot** the points obtained in (b)-(c) and make use of (a) to **sketch the graph** of **C** in the following polar chart**:** [*Indicate **important values** of* r and θ *in the **outer circle of the chart***]



Part II (8 MCQ: 6pts/each**)** **Encircle** your **Choice** for each MCQ on the **front page** of your answer book**)**

Q1. If the end points of a diameter of a sphere lie at $A(1,4,-2)$ and $B(-7,1,2)$ then an equation of the sphere is given by

(a^*) $x^2+y^2+z^2+6x-5y=7$

(b) $x^2+y^2+z^2-8x-4y=10$

(c) $x^2+y^2-z^2+6x-4y=7$

(d) $x^2+y^2+z^2+7x-10y=20$

(e) $x^2+y^2+z^2+6x+4y=12$

Sol: Mid point is $\left(\frac{1-7}{2},\frac{4+1}{2},\frac{-2+2}{2}\right)=\left(-3,\frac{5}{2},0\right)$

and radius is $R=\sqrt{(1+3)^2+\left(4-\frac{5}{2}\right)^2+(-2-0)^2}=\sqrt{16+\frac{9}{4}+4}=\frac{\sqrt{89}}{2}$

equation of the sphere is

$(x+3)^2+\left(y-\frac{5}{2}\right)^2+z^2=\frac{89}{4}$

$x^2+y^2+z^2+6x-5y=\frac{89}{4}-9-\frac{25}{4}=\frac{89-36-25}{4}=\frac{89-61}{4}=\frac{28}{4}=7$

Q2. Suppose that a 3-D vector $\vec{v}$ lies below the *xy*-plane and has the **direction angles** $\alpha,\ \beta,\ \gamma$ with *x*, *y* and *z* axes respectively. If $\alpha=\frac{\pi}{4},\ \beta=\frac{\pi}{3}$, then the value of γ is given by

(a^*) $2\pi/3$

(b) $(\sqrt{2}\,\pi)/2$

(c) $-1/2$

(d) $5\pi/6$

(e) $-1/\sqrt{2}$

Sol: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \implies \frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1 \implies \cos^2\gamma=1-\frac{3}{4}=\frac{1}{2}$

Since the vector $\vec{v}$ lies below *xy*-plane, therefore, $\cos\gamma=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2} \implies \gamma=\frac{2\pi}{3}$.

Q3. A **value of** α for which the vectors $\vec{u}=3\vec{i}+\alpha\,\vec{k}$ and $\vec{v}=2\alpha\,\vec{i}-\vec{j}$ have the **same length** is given by

(a^*) $\sqrt{8/3}$

(b) $\sqrt{5/3}$

(c) $\sqrt{8/5}$

(d) $\sqrt{7/3}$

(e) $\sqrt{5/8}$

Sol: $3^2+\alpha^2=4\alpha^2+1 \implies 3\alpha^2=8 \implies \alpha=\sqrt{\frac{8}{3}}$.

Q4. The area of the triangle with the vertices $(a,0,0)$, $(0,2a,0)$ and $(0,0,3a)$ is

(a^*) $7a^2/2$

(b) $5a^2/2$

(c) $6a^3$

(d) $7a$

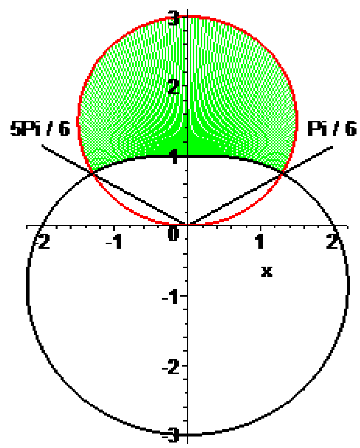
(e) $3a^3/2$

Sol: Let $\vec{u} = \langle -a, 2a, 0\rangle$ *and* $\vec{u} = \langle -a, 0{,}3a\rangle$ be two adjacent vectors. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a & 2a & 0 \\ -a & 0 & 3a \end{vmatrix} = 6a^2\hat{\imath} + 3a^2\hat{\jmath} + 2a^2\hat{k}$$

and area $A = \frac{1}{2}|\vec{u} \times \vec{v}| = \frac{1}{2}\sqrt{36a^4 + 9a^4 + 4a^4} = \frac{1}{2}\sqrt{49a^4} = \frac{7a^2}{2}$

Q5. The area of the region inside the curve $r = 3\sin\theta$ and outside the curve $r = 2 - \sin\theta$ is



(a^*) $\int_{\pi/6}^{5\pi/6} (4\sin^2\theta + 2\sin\theta - 2)d\theta$

(b) $\int_{-\pi/6}^{\pi/6} (2\sin^2\theta - 2\sin\theta - 1)d\theta$

(c) $\int_{\pi/6}^{5\pi/6} (4\sin^2\theta + \sin\theta + 3)d\theta$

(d) $\int_{\pi/6}^{5\pi/6} (4\sin^2\theta + 5\sin\theta - 2)d\theta$

(e) $\int_{-\pi/6}^{\pi/6} (4\sin^2\theta + 5\sin\theta - 2)d\theta$

Sol: $A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (9\sin^2\theta - (2 - \sin\theta)^2)d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta \right)d\theta$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta + 4\sin\theta - 4 \right)d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4\sin^2\theta + 2\sin\theta - 2 \right)d\theta$

Q6. The Cartesian equation of the curve $x = \ln t,\ y = \sqrt{t},\ t \geq 1$ is given by

$(a^*)\ \ y = e^{x/2},\ \ x \geq 0$

$(b)\ \ y = e^{x},\ \ x \geq 1$

$(c)\ \ y = e^{x/2},\ \ x \geq 1$

$(d)\ \ y = e^{x},\ \ x \geq 0$

$(e)\ \ y = e^{2x},\ \ x \geq 0$

Sol: $x = \ln t\,, t \geq 0 \implies t = e^{x}, x \geq 0$

$$y = \sqrt{t} = e^{\frac{x}{2}}, x \geq 0.$$

Q7. The slope of the tangent line to the polar curve $r = \cos\theta + 1$ at $\theta = \pi/2$ is

$(a^*)\ \ 1$

$(b)\ \ 1/2$

$(c)\ \ 1/3$

$(d)\ \ 0$

$(e)\ \ -1/2$

Sol: $x = r cos\theta = cos\theta + cos^2\theta,\ \ \frac{dx}{d\theta} = -sin\theta - 2sin\theta\ cos\theta$

$y = r sin\theta = sin\theta + sin\theta\ cos\theta,\ \ \frac{dy}{d\theta} = cos\theta + cos2\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{cos\theta + cos2\theta}{-sin\theta - 2sin\theta\ cos\theta}, \quad \text{and at } \theta = \frac{\pi}{2},\ m = \frac{dy}{dx} = \frac{-1}{-1} = 1\ .$$

Q8. Two **forces** ***F*** and **G** are acting on an object placed at the **origin** of the ***xy*-plane** with **magnitudes** 1 N and 2 N respectively.

If ***F*** acts along the **positive** *y-axis* and **G** makes an **angle** of $\theta = \pi/3$ with the **positive** *x*-axis, then the **magnitude** of the **resultant** force ***F + G*** is

$(a^*)\ \ \sqrt{5 + 2\sqrt{3}}$ N

$(b)\ \ \sqrt{1 + \sqrt{3}}$ N

$(c)\ \ \sqrt{2 + \sqrt{3}}$ N

$(d)\ \ 5 + \sqrt{3}/2$ N

$(e)\ \ \sqrt{2 + 2\sqrt{3}}$ N

Sol: $\vec{F} = 0\hat{\imath} + 1\hat{\jmath} = \hat{\jmath}$ and $\vec{G} = 2cos\frac{\pi}{3}\hat{\imath} + 2sin\frac{\pi}{3}\hat{\jmath} = \hat{\imath} + \sqrt{3}\hat{\jmath}$

$\vec{F} + \vec{G} = \hat{\imath} + (1 + \sqrt{3})\hat{\jmath}$ and $\left|\vec{F} + \vec{G}\right| = \sqrt{1 + (1 + \sqrt{3})^2} = \sqrt{1 + 1 + 2\sqrt{3} + 3} = \sqrt{5 + 2\sqrt{3}}$

KEY for 4 versions of Part II (MCQ's)

Q #	V. **I**		Q #	V. **II**		Q #	V. **III**		Q #	V. **IV**
1	***b***		1	***e***		1	***d***		1	***c***
2	***d***		2	***a***		2	***b***		2	***e***
3	***d***		3	***e***		3	***a***		3	***b***
4	***c***		4	***b***		4	***e***		4	***a***
5	***e***		5	***d***		5	***c***		5	***a***
6	***a***		6	***c***		6	***b***		6	***e***
7	***e***		7	***b***		7	***e***		7	***c***
8	***b***		8	***a***		8	***e***		8	***e***