King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2011-2012 (111)

Tuesday, October 11, 2011. Allowed Time: 2 hours

Name:____

ID Number: __________________________ Serial Number:_________

Section Number:__________ Instructor's Name:____________________

Instructions

1. Write neatly and eligibly. You may lose points for messy work.
2. Show all your work. No points for answers without justification.
3. Calculators and Mobiles are not allowed.
4. Make sure that you have 10 different problems (7 pages + cover page).

Question	Points	Your Score
Q1	14	
Q2	11	
Q3	11	
Q4	11	
Q5	11	
Q6	7	
Q7	8	
Q8	7	
Q9	10	
Q10	10	
TOTAL	100	

1. The graph of f is given.

(a) (8 points) Evaluate the limit if it exists. If it does not exist, explain why. Use the symbols ∞ or $-\infty$ as appropriate.

1 i. $\lim_{x \to 2^+} f(x) = 3$

1 ii. $\lim_{x \to -3^+} f(x) = 0$

1 iii. $\lim_{x \to -3} f(x)$ DNE, since $\lim_{x \to -3^-} f(x) = -2 \neq \lim_{x \to -3^+} f(x)$

1 iv. $\lim_{x \to 4} f(x) = 2$

1 v. $\lim_{x \to 0} f(x) = \infty$

1 vi. $\lim_{x \to 2^-} f(x) = -\infty$ (DNE is not acceptable)

1 vii. $\lim_{x \to \infty} f(x) = 4$

1 viii. $\lim_{x \to -\infty} f(x) = -1$

(b) (2 points) State the equations of the horizontal asymptotes.

1 1

$y = -1, \quad y = 4$

(c) (2 points) State the equations of the vertical asymptotes.

1 1

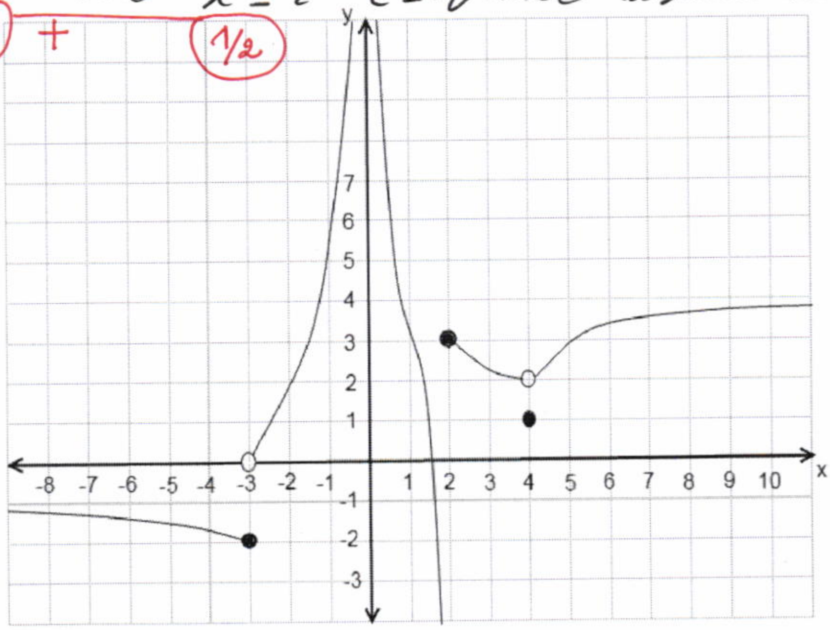
$x = 0, \quad x = 2$

(d) (2 points) At what numbers f is discontinuous? Explain.

$\frac{1}{2}$ At $x = -3$ (Jump discontinuity)

1 At $x = 0$ and $x = 2$ (infinite discontinuity)

1/2 + 1/2



$\frac{1}{2}$ At $x = 4$ (removable discontinuity)

2. Evaluate the limit, if it exists:

(a) (2 points) $\lim_{x\to 1}\left(\dfrac{\frac{x}{2}+\frac{2}{x}}{2+x}\right)$.

$$= \frac{\frac{1}{2}+2}{3} = \frac{5}{6}$$

(b) (3 points) $\lim_{x\to -3^-}\left(\dfrac{x^2-9}{[[x+3]]}\right)$, where [[x]] denotes the greatest integer less than or equal to x.

As $x \to -3^-$, then $x+3 \to 0^-$. Thus $[[x+3]] \to -1$

So $$\lim_{x\to -3^-}\frac{x^2-9}{[[x+3]]} = \frac{0}{-1} = 0$$

(c) (3 points) $\lim_{x\to 2^-}\dfrac{2-\sqrt{7x-10}}{x-2}$.

$$= \lim_{x\to 2^-}\frac{4-(7x-10)}{(x-2)\left[2+\sqrt{7x-10}\right]}$$

$$= \lim_{x\to 2^-}\frac{-7(x-2)}{(x-2)\left[2+\sqrt{7x-10}\right]}$$

$$= \lim_{x\to 2^-}\frac{-7}{2+\sqrt{7x-10}} = \frac{-7}{2+2} = -\frac{7}{4}$$

(d) (3 points) $\lim_{x\to\infty}\dfrac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$.

$$= \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}\cdot\frac{x}{x}$$

$$= \lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{1}{x}-\frac{1}{x^2}} = \infty$$

Or

$$= \lim_{x\to\infty}\frac{\frac{x^3-1}{x^4}}{\frac{x-1}{x^3}}$$

$$= \lim_{x\to\infty}\frac{x^3-1}{x(x-1)}$$

$$= \infty$$

3. Evaluate the limit, if it exists:

(a) (3 points) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$.

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)} \quad (1)$$

$$= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x}{x\left[\sqrt{1+\frac{1}{x}}+1\right]} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \ (1) = \frac{1}{2} \quad (1)$$

(b) (3 points) $\lim_{x\to 0}\left(4+x+x^2\cdot\sin\frac{\pi}{x}\right)$.

$$-1 \le \sin\frac{\pi}{x} \le 1 \quad , x \ne 0$$

x^2 always positive.

$$-x^2 \le x^2 \sin\frac{\pi}{x} \le x^2$$

$$4+x-x^2 \le 4+x+x^2\sin\frac{\pi}{x} \le 4+x+x^2 \quad (1)$$

since $\lim_{x\to 0} 4+x-x^2 = \lim_{x\to 0} 4+x+x^2 = 4$ (1)

Then by using squeez theorem

$$\lim_{x\to 0}\left(x^2\sin\frac{\pi}{x}+4+x\right) = 4 \quad (1)$$

(c) (3 points) $\lim_{x\to 0}\frac{\sin^2 x}{\cos x-1}$.

$$(1) = \lim_{x\to 0}\frac{1-\cos^2 x}{\cos x-1}$$

$$(1) = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{\cos x-1}$$

$$(1) = \lim_{x\to 0}\frac{1+\cos x}{-1} = -2$$

Or

$$(1) = \lim_{x\to 0}\frac{\sin^2 x}{\cos x-1}\cdot\frac{\cos x+1}{\cos x+1}$$

$$= \lim_{x\to 0}\frac{\sin^2 x(\cos x+1)}{\cos^2 x-1}$$

$$(1) = \lim_{x\to 0}\frac{\sin^2 x(\cos x+1)}{-\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\cos x+1}{-1}$$

$$(1) = -2$$

(d) (2 points) $\lim_{x\to\frac{\pi}{2}^+}\arctan\left(\frac{x}{\sin 2x}\right)$.

As $x \longrightarrow \frac{\pi}{2}^+$, then $\frac{x}{\sin 2x} \longrightarrow -\infty$ (1)

So $\lim_{x\to\frac{\pi}{2}^+} \arctan\left(\frac{x}{\sin 2x}\right) = -\frac{\pi}{2}$ (1)

4. Let

$$f(x) = \begin{cases} \dfrac{6\,c^2}{x+1} , & \text{if } x > 1 \\ 27 , & \text{if } x = 1 \\ c^3\,x , & \text{if } x < 1 \end{cases}$$

(a) (6 points) Find the values of c so that $f(x)$ is continuous everywhere.

$\lim\limits_{x \to 1^+} f(x) = 3c^2$ 2 , $\lim\limits_{x \to 1^-} f(x) = +c^3$ 2

$f(x)$ is continuous at $x=1$, iff $f(1) = 3c^2 = +c^3 = 27$ 1

then $c = 3$ 1

(b) (5 points) Find the values of c so that $f(x)$ has removable discontinuity.

$f(x)$ has a removable discontinuity at $x=1$, iff

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x)$ but $f(1) \neq \lim\limits_{x \to 1} f(x)$ 3

then $c = 0$ 2

5. Let $f(x) = \dfrac{|x-1|}{x^2(x-1)}$.

(a) (6 points) Use limits to find all vertical asymptotes of the graph of f.
(Justify your answer)

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{x-1}{x^2(x-1)} = \lim\limits_{x \to 1^+} \dfrac{1}{x^2} = 1$ 1

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{1-x}{x^2(x-1)} = \lim\limits_{x \to 1^-} \dfrac{-1}{x^2} = -1$ 1

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{-(x-1)}{x^2(x-1)} = \lim\limits_{x \to 0^+} \dfrac{-1}{x^2} = -\infty$ 2

$\therefore$ $x = 0$ is a vertical asymptote 2

(b) (5 points) Use limits to find all horizontal asymptotes of the graph of f.
(Justify your answer)

$\lim\limits_{x \to \infty} \dfrac{x-1}{x^2(x-1)}$ 1 $= \lim\limits_{x \to \infty}$ 1 $\dfrac{1}{x^2} = 0$

$\lim\limits_{x \to -\infty} \dfrac{1-x}{x^2(x-1)}$ 1 $= \lim\limits_{x \to -\infty}$ 1 $\dfrac{-1}{x^2} = 0$

$\therefore$ $y = 0$ 1 is a horizontal asymptote.

6. Let $f(x) = \dfrac{1}{x+1}$.

(a) (4 points) Use limits to find the slope of the tangent line to the graph of f at the point $P(0,1)$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \; ② = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h}$$

$$= \lim_{h\to 0} \frac{1-(h+1)}{h(h+1)} = \lim_{h\to 0} \frac{-h}{h(h+1)} \quad ①$$

$$= \lim_{h\to 0} \frac{-1}{h+1} = -1 \quad ①$$

(b) (3 points) Find the equation of the tangent line to the graph of f at the point $P(0,1)$.

$$y - 1 = -1(x-0) \quad ②$$

$$y - 1 = -x$$

$$y = 1 - x \quad ①$$

7. (8 points) Using the $\epsilon - \delta$ definition of limit, prove that $\lim_{x\to 0}(1-2x) = 1$.

1. Guessing a value for δ:

Let $\epsilon > 0$. We want to find $\delta > 0$ s.t.

$|f(x) - 1| < \epsilon$ ① whenever $0 < |x - 0| < \delta$ ①

But $|1 - 2x - 1| < \epsilon$ ① $\Rightarrow |-2x| < \epsilon$ ①

$\Rightarrow 2|x| < \epsilon$ ①

$\Rightarrow |x| < \frac{\epsilon}{2}$ ①

So we should choose $\delta = \frac{\epsilon}{2}$ ①

2. Showing that δ works:

$\forall \epsilon > 0, \exists \delta = \frac{\epsilon}{2} > 0$ s.t.

$0 < |x - 0| < \frac{\epsilon}{2} \Rightarrow 0 < |x| < \frac{\epsilon}{2}$

$\Rightarrow 0 < |2x| < \epsilon$

$\Rightarrow |(1-2x) - 1| < \epsilon$ ①

Note:
If the student uses ($\Leftrightarrow$) instead of ($\Rightarrow$) to find $\delta = \frac{\epsilon}{2}$, he may get full mark.

8. The displacement (in meters) of a particle moving in a straight line is given by $s(t) = 2\,t^2 + 5$, where t is measured in seconds.

(a) (4 points) Find the average velocity over the interval $[\,1, 1+h]$.

$$V_{av} = \frac{S(1+h) - S(1)}{(1+h) - 1} \quad 2 \quad = \frac{2(1+h)^2 + 5 - 7}{h} \quad 1$$

$$= \frac{2(1+2h+h^2) - 2}{h} = \frac{2h(2+h)}{h}$$

$$= 2(2+h) \quad 1$$

(b) (3 points) Use part (a) to find the instantaneous velocity v when $t = 1$.

$$V = \lim_{h \to 0} \frac{S(1+h) - S(1)}{h} \quad 2$$

$$= \lim_{h \to 0} 2(2+h) = 4 \quad 1$$

9. (10 points) Use the Intermediate Value Theorem to show that the equation $\cos x = x^2$ has at least two real roots in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $f(x) = \cos x - x^2$ 1

$f(\frac{\pi}{2}) = -\frac{\pi^2}{4}$ 1

$f(-\frac{\pi}{2}) = -\frac{\pi^2}{4}$ 1

but $f(0) = 1$ 1

So, we can apply (I.V.T.) on $[-\frac{\pi}{2}, 0]$ and $[0, \frac{\pi}{2}]$

ⓐ On $[-\frac{\pi}{2}, 0]$

① $f(x)$ is cont. on $[-\frac{\pi}{2}, 0]$ 1

② $f(-\frac{\pi}{2}) \neq f(0)$ and

$f(-\frac{\pi}{2}) < N=0 < f(0)$ 1

Then (by using I.V.T.),

$\exists c \in (-\frac{\pi}{2}, 0)$ s.t.

$f(c)=0$ or $\cos c = c^2$ 1

ⓑ On $[0, \frac{\pi}{2}]$

① $f(x)$ is cont. on $[0, \frac{\pi}{2}]$ 1

② $f(0) \neq f(\frac{\pi}{2})$ and

$f(\frac{\pi}{2}) < N=0 < f(0)$ 1

Then (by using I.V.T.),

$\exists c \in (0, \frac{\pi}{2})$ s.t.

$f(c)=0$ or $\cos c = c^2$ 1

10. (10 points) The tangent line T to the graph of $y = f(x)$ at the point $P(a, 3)$ passes through the point $(4, -9)$. If $f'(a) = 6$, find the equation of the line perpendicular to T at P.

[write your answer in the form $y - y_1 = m(x - x_1)$]

slope of the tangent at $P = \dfrac{3-(-9)}{a-4} = f'(a) = 6$ 3

$\Rightarrow \dfrac{12}{a-4} = 6 \Rightarrow a - 4 = 2 \Rightarrow a = 6$ 2

The slope of the line ~~pe~~perpendicular to T at P.

$= -\dfrac{1}{f'(a)} = -\dfrac{1}{6}$ 2

So the equation of the line perpendicular to T at P is

$y - \underset{1}{3} = \underset{1}{-\dfrac{1}{6}}(x - \underset{1}{6})$