Math 590/690 (101)

Applied Fractional Calculus

Midterm Exam

(1) Show that if $f \in AC[a,b]$ then $f = I_a^\alpha D_a^\alpha f$ for all $0 < \alpha < 1$. What happens if $\alpha > 1$?

(2) Consider the equation

$$D_0^{\sqrt{2}} u(t) = 4, \quad t > 0.$$

(a) Find the general solution.

(b) Find the general solution which is bounded at $t = 0$.

(3) Define the space of functions:

$$S^\alpha(a,b) = \left\{ f \in L^1(a,b) : \quad I_a^{n-\alpha} f \in AC^n[a,b] \right\}, \quad n = -[-\alpha], \quad \alpha > 0.$$

A function $f \in S^\alpha(a,b)$ is said to have a summable fractional derivative.

Consider the function

$$f(t) = \begin{cases} 1, & 0 \le t \le 2, \\ 0, & 2 < t < T. \end{cases}$$

(a) Show that $f(t) \in S^\alpha(0,T)$ for $0 < \alpha < 1$.

(b) Calculate $I_0^\alpha D_0^\alpha f(t)$ for $0 < \alpha < 1$.

(4) Use Formula (2.202) to evaluate $D_2^{1/2} \left(t\sqrt{t-2} \right)$.

(5) Verify Formula (2.211) for the function $K(t,\tau) = (t-\tau)^{-1/2}$.

(6) Show that the limit in (2.217) is not true in general.
