KFUPM, Department of Mathematics and Statistics

Final Exam for MATH 572, Semester 101

Problem 1 Consider the following problem

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in } \Omega \times (0,T] \\ u_x(0,t) = u(1,t) = 0 \quad \text{for } t \in (0,T) \\ u(x,0) = v(x) \quad u_t(x,0) = w(x) \quad \text{for } x \in \Omega \end{cases} \tag{1}$$

where $\Omega = (0,1)$ and $u = u(x,t)$.

a) Define the spatial piecewise linear finite element solution u_h

b) Assume that $u_h(x,0) = v_h(x)$ and $u_{h_t}(x,0) = w_h(x)$. Derive the following stability property of u_h:

$$\|u_{h_t}(t)\| + \|u_{h_x}(t)\| \le \|v_h\| + \|w_{h_x}\| \quad \text{for each } t \in (0,T]$$

c) Show that for each $t \in (0,T]$, the finite element solution u_h is unique.

d) State the expected order of convergence of $u_h(t)$ to $u(t)$ in the spatial H^1-norm (without proof)

Problem 2 Consider the following two-point BVP:

$$-a(x)\,u'' + u = f(x) \quad \text{in } (0,1) \quad \text{with } u(0) = u(1) = 0. \tag{2}$$

where $0 < c_0 \le a(x) \le c_1$ for all $0 \le x \le 1$.

a) Define the cubic Petrov Galerkin (PG) solution $\tilde{u}_h$ for (2)
b) Derive the following stability property of $\tilde{u}_h$: $\|\tilde{u}_h''\| \le C\|f\|$
c) Prove that $\tilde{u}_h$ exists and is unique
d) Show that $\|u - \tilde{u}_h\|_2 \le Ch^2\|u\|_4$

Hint for part (d): You may need to use the following. Say that $\tilde{u}_h \in \tilde{S}_h$ for some finite dimensional space $\tilde{S}_h$ need to be defined in part (a). Assume that there exists $\widehat{u} \in \tilde{S}_h$ such that $\|\widehat{u} - u\|_\ell \le Ch^{4-\ell}\|u\|_4$. for $\ell = 0, 1, 2$.

Problem 3 Consider the following one dimensional parabolic problem

$$\begin{cases} u_t - u_{xx} = \sin u \quad \text{in } (0,1)\times(0,T) \\ u(0,t) = u(1,t) = 0 \quad \text{for } t \in (0,T) \\ u(x,0) = v(x) \quad \text{for } x \in (0,1) \end{cases} \tag{3}$$

where $u = u(x,t)$. Assume that v is a sufficiently regular function.

a) Define the Backward Euler scheme in time

b) Show the stability of the proposed scheme

c) Show that this scheme is first order accurate

Problem 4 Consider the following two-point BVP:

$$-a(x)\,u'' = f(x) \quad \text{in } (0,1) \quad \text{with } u(0) = b \quad \text{and} \quad u(1) = c \tag{4}$$

where $0 < c_0 \leq a(x) \leq c_1$ for all $0 \leq x \leq 1$.

a) Define the cubic collocation method (CM) for (4)
b) Say that, u_h is the approximate solution obtained from the CM. Prove that u_h is unique.
c) Does the existence of u_h follow from the uniqueness? Justify your answer
d) State the expected error $\|u - u_h\|$ (without proof)
e) State one advantage and one disadvantage of the CM over the standard Galerkin method