King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 568 - Final Exam (101)   Time: 2 hours 30

Jan. 26, 2011

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Name: ID #

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Problem 1	/10
Problem 2	/10
Problem 3	/10
Problem 4	/10
Problem 5	/5
Problem 6	/5
Problem 7	/5
Total	/55

Problem # 1. (10 marks) Solve the problem

$$
\begin{aligned}
&u_{tt} - \Delta u = 0, \qquad (x, y) \in \mathbb{R}^2, \quad t > 0 \\
&u(x, y, 0) = x^2 + y^2, \quad u_t(x, y, 0) = xy
\end{aligned}
$$

Problem # 2. (10 marks) Let Ω be a bounded and a smooth domain of $\mathbb{R}^n$ and $\phi \in C^1(\overline{\Omega})$. Given the problem

$$(P_1) \quad \begin{array}{lll} u_{tt} - \Delta u + \nabla\phi \cdot \nabla u = 0, & x \in \Omega, \quad t > 0 & \\ u(x,t) = 0, & \text{on } \partial\Omega, \quad t \geq 0 & \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & & (x,y) \in \Omega \end{array}$$

a) Show that the solution of (P_1) satisfies

$$\int_\Omega e^{-\phi(x)} \left[u_t^2(x,t) + |\nabla u(x,t)|^2\right] dx = \int_\Omega e^{-\phi(x)} \left[u_1^2(x) + |\nabla u_0(x)|^2\right] dx, \quad \forall t \geq 0$$

Hint: Multiply by $e^{-\phi(x)} u_t$ and integrate
b) Show that (P_1) has at most one solution.

Problem # 3. (10 marks) Given the problem

$$(P_2)\quad \begin{array}{ll} u_t(x,t) - u_{xx}(x,t) + u(x,t) = 0, & -\infty < x < +\infty, \quad t > 0 \\ u(x,0) = x, \quad -\infty < x < +\infty & \end{array}$$

a. Find the solution u of (P_2)

b. Find $\lim_{t\to\infty} u(x,t)$

c. Can you deduce, from (a), the solution of

$$\begin{array}{ll} u_t - \Delta u + u = 0, & (x,y,z) \in \mathbb{R}^3, \quad t > 0 \\ u(x,y,z,0) = xyz & \end{array}$$

Problem # 4. (10 marks) Solve the problem

$$\begin{aligned}
&u_t(x,t) - u_{xx}(x,t) = t\cos x, \qquad 0 < x < \pi, \quad t > 0 \\
&u_x(0,t) = u_x(\pi,t) = 0, \qquad t \geq 0 \\
&u(x,0) = 2 + \cos 3x + \cos x
\end{aligned}$$

Problem # 5. (5 marks) Given the problem

$$\begin{aligned} \Delta u(x,y) &= 0, && \text{in } \Omega \\ u(x,y) &= xy, && \text{on } \partial\Omega \end{aligned} \tag{0.1}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4, \}$. Show that

$$-2 \le u(x,y) \le 2, \qquad \forall (x,y) \in \Omega$$

Problem # 6. (5 marks) Does the following problem have solution ?

$$\begin{array}{ll} \Delta u(x,y) = xy, & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 2, & \text{on } \partial\Omega \end{array} \tag{0.2}$$

where Ω is the triangle whose vertices are $(0,0)$, $(0,1)$, and $(1,0)$ and η is the unit outer normal to Ω. (justify your answer)

Problem # 7. (5 marks) Given the problem

$$\Delta u(x,y) = f(x,y), \qquad \text{in } \mathbb{R}^2 \tag{0.3}$$

where $f(x,y) = \begin{cases} 1 - \sqrt{x^2+y^2}, & \text{for } x^2+y^2 \le 1 \\ 0, & \text{for } x^2+y^2 > 1 \end{cases}$.

Find $u(0,0)$.