King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math550 "Linear Algebra"

Semester 101 (Fall 2010)

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Final Exam (24.1.2011) **Duration**: 3 hours

Remark: Give *self-contained* proofs and arguments. Work out all details of examples/counterexamples.

Part I: Solve any 2 of Q1-Q3.

Q1. (15 points) Let V be an inner product space. A linear operator $T : V \longrightarrow V$ is said to be an *isometry*, or a *distance-preserving transformation*, iff

$$\|T(\alpha)\| = \|\alpha\| \text{ for every } \alpha \in V.$$

(a) Show that $T \in \mathcal{L}(V, V)$ is an isometry if and only if

$$< T(\beta) \mid T(\gamma) >=< \beta \mid \gamma > \text{ for all } \beta, \gamma \in V.$$

(b) Provide two *types* of isometries on $V = \mathbb{R}^2$ and interpret them geometrically.

Q2. (15 points) An $n \times n$ real matrix $A = (a_{ij})$ is said to be a *Markov matrix* iff

$$a_{ij} \geq 0 \text{ for all } 1 \leq i, j \leq n \text{ and } \sum_{j=1}^{n} a_{ij} = 1 \text{ for all } 1 \leq i \leq n.$$

Show that
(a) If A and B are Markov matrices, then AB is a Markov matrix.
(b) If λ is a characteristic value of a Markov matrix, then $|\lambda| \leq 1$.
(c) If $a_{ij} > 0$ for all $1 \leq i, j \leq n$, then $\text{nullity}(A - I) = 1$.

Q3. (15 points) Let V be an n-dimensional vector space and $S, T : V \longrightarrow V$ be linear transformations.
(a) Show that $\text{nullity}(S + T) \geq \text{nullity}(S) + \text{nullity}(T) - n$.
(b) Are there non-zero linear transformations S, T for which we have equality in (a)?
(c) Show that $\max\{\text{nullity}(S), \text{nullity}(T)\} \leq \text{nullity}(ST) \leq \text{nullity}(S) + \text{nullity}(T)$.

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Part II: Prove any 2 of the theorems in Q4-Q6.

Let V be a finite dimensional vector space over the field $\mathbb{F}$ and $T : V \longrightarrow V$ a linear operator with characteristic polynomial $f(x)$ and minimal polynomial $p(x)$.

Q4. (15 points) T is trinagularble if and only if $p(x)$ is a product of linear polynomials over $\mathbb{F}$.

Q5. (15 points) T is diagonalizable if and only if $p(x)$ is a product of distinct linear polynomials over $\mathbb{F}$.

Q6. (15 points) $p(x)$ divides $f(x)$ and both have the same prime factors in $\mathbb{F}[x]$.

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Part III: Solve each of Q7 & Q8:

Q7. (20 points) Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

a) Show that A is triangulable but not diagonalizable.
b) Find the Jordan Canonical Form of A.
c) Find an invertible matrix P such that $P^{-1}AP$ is in JCF.

Q8. (20 points) Prove or disprove each of the following statements:

(a) There exists a real 3×3 matrix A such that $A^2 = -I$.

(b) There exists a matrix A such that $\text{ch}(A) = (x-3)^8$, $\min(A) = (x-3)^4$ and $\text{nullity}(A-3I) = 3 = \text{nullity}(A-3I)^2$.

(c) If $\{\alpha_1, \cdots, \alpha_n\}$ is any orthonormal basis for $\mathbb{R}^n$, then for any β we have

$$\sum_{i=1}^{k} < \beta, \alpha_i > \leq ||\beta||^2 \text{ for each } 1 \leq k \leq n.$$

(d) If $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous, then

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \left(\int_a^b f^2(x)dx \right) \left(\int_a^b g^2(x)dx \right).$$

e) If V is an inner product space and $W \leq V$ is a subspace, then $W = (W^{\perp})^{\perp}$.

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Part IV: Bonus (10 points)

Give an example of an inner product space V and two subspaces $W_1, W_2 \leq V$ such that

$$W_1^{\perp} + W_2^{\perp} \subsetneqq (W_1 \cap W_2)^{\perp}.$$

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