King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math550 "Linear Algebra"

Semester 101 (Fall 2010)

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First Major Exam (6.11.2010)

Remark: Give *self-contained* proofs and arguments. Work out all details of examples/counterexamples.

Part I: Solve any 2 of Q1-Q3.

Q1. (15 points) Let V be an n-dimensional vector space over the field $\mathbb{F}$. Define the *index* of a linear transformation $T : V \longrightarrow V$ to be the smallest nonnegative integer k such that $T^k(V) = T^{k+1}(V)$. Show that

(a) If T is invertible, then $\text{index}(T) = 0$ (notice that $T^0 := I$).

(b) If T is nilpotent, then $\text{index}(T)$ is the smallest positive integer k such that that $T^k = 0$.

Q2. (15 points) Let V be a vector space over $\mathbb{R}$. A subset $C \subseteq V$ is said to be *convex* iff for any $v, w \in C$, the set C contains also also the *line segment*

$$L = \{(1-t)v + tw \mid 0 \leq t \leq 1\} \tag{1}$$

Let $A = \{\alpha_1, \cdots, \alpha_n\} \subseteq V$ and consider

$$C(A) := \{c_1\alpha_1 + \cdots + c_n\alpha_n \mid c_i \geq 0 \text{ and } c_1 + \cdots + c_n = 1\}.$$

(a) Show that $C(A)$ is convex.

(b) Prove that $C(A)$ is contained in any convex subset $S \subseteq V$ that contains A. (**Hint:** Use mathematical induction on n).

(c) If $V = \mathbb{R}^2$ and $A = \{(1,2), (-1,3), (0,0)\}$ determine $C(A)$ geometrically.

Q3. (15 points) Let $\mathbb{M}_n(\mathbb{F})$ be the vector space of all $n \times n$ matrices with entries in the field $\mathbb{F}$. The *trace* functional is

$$\text{trace} : \mathbb{M}_n(\mathbb{F}) \longrightarrow \mathbb{F},\ X \mapsto \sum_{i=1}^{n} X_{ii}$$

Let $A, B \in \mathbb{M}_n(\mathbb{F})$.

(a) Show that $\text{trace}(AB) = \text{trace}(BA)$.

(b) Prove that if A is similar to B, then $\text{trace}(A) = \text{trace}(B)$. Disprove the converse.

(c) Can it happen that $AB - BA = I_n$?

Part II: Prove any 2 of the theorems in Q4-Q6.

Q4. (15 points) Let V and W be vector spaces over the field $\mathbb{F}$ and $T : V \to W$ is a linear transformation. If V is finite dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim V. \tag{2}$$

Q5. (15 points) Show that if A is an $m \times n$ matrix with entries in the field $\mathbb{F}$, then

$$\text{row rank}(A) = \text{column rank}(A).$$

Q6. (15 points) Let V be a finite dimensional vector space over the field $\mathbb{F}$. Show that for any vector subspace $W \leq_{\mathbb{F}} V$ we have

$$\dim W + \dim W^0 = \dim V.$$

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Part III: Solve each of Q7 & Q8:

Q7. (20 points) Let W be the solution space of

$$\begin{aligned} x + 2y + z &= 0 \\ -x + y - 3z &= 0 \\ 3x + 3y + 5z &= 0 \end{aligned}$$

(a) Find a basis β for W.
(b) Extend β to a basis for $V = \mathbb{R}^3$.
(c) Find a dual basis for V^*.
(d) Find a basis for W^0.

Q8. (20 points) State whether the following statements are TRUE or FALSE. If a statement is FALSE, provide a counterexample.

(a) For any vector space V, we have $V \simeq V^{**}$.
(b) Every square matrix can be written in a *unique* way as a sum of a symmetric matrix and a skew-symmetric matrix.
(c) If a linear transformation T is idempotent, then $T - 2I$ is invertible.
(d) If A, B are two $n \times n$ matrices, then

$$\text{rank}(AB) \geq \max\{\text{rank}(A), \text{rank}(B)\}.$$

(e) If a linear transformation T satisfies $T^2 - T - 2I = 0$, then $T = 2I$ or $T = -I$.

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Part IV: Bonus (10 points)

Show that $\mathbb{R}[x]^* \simeq \mathbb{R}[[x]]$ as real vector spaces.

GOOD LUCK