King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH-480 Linear and Nonlinear Programming [1]
Solution Major Exam 2

Exercice 1 **(**35 **pts)**

Given the following pair of linear programs:

$$\begin{array}{llll} \max\limits_{x,y} & z = x + 2y & \min\limits_{\alpha,\beta} & \gamma = \alpha + 3\beta \\ \text{s.t} & y \leq \frac{1}{3}, & \text{s.t} & \beta \geq 1 \\ & x + y \leq 1, & & \alpha + \beta \geq 2 \\ & x, y \geq 0 & & \alpha, \beta \geq 0. \end{array}$$

(a) Solve each linear program graphically.(10pts)
The optimal solution the first program is: $x^* = \frac{2}{3}$, $y^* = \frac{1}{3}$ *and the the optimal objective is* $z* = \frac{4}{3}$. *The optimal solution the second program is:* $\alpha^* = 1$, $\beta^* = 1$ *and the the optimal objective is* $\gamma* = 4$.

(b) Is there any relation between these two linear programs. (5pts)
There is a Primal-Dual relation between the two linear programs.

(c) Write the complementary slackness conditions corresponding to these two linear programs. (10pts)

The complementary slackness conditions corresponding to the two linear programs can be expressed:

$$(\alpha, \beta)\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(d) Propose a method to linearize these complementary slackness conditions. (10pts)

To linearize these conditions we introduce the binary vector of variables u *and two real parameters* K *and* L *arbitrarily chosen with* $K < 0$ *and* $L > 0$.

Conditions written in (c) can then be written as follows:

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}\right] \geq K\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

[1]Dr. Slim Belhaiza (c), 22 December, 2010

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}\right] \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq L(\mathbb{1} - u),$$

$$\begin{aligned} &\beta && \geq 1 \\ &\alpha + \beta && \geq 2 \\ &x, y && \geq 0, \\ &\alpha, \beta && \geq 0. \end{aligned}$$

Exercice 2

Consider the following linear program:

$$\begin{array}{ll} \max\limits_{x_1,x_2,x_3} & 3x_1 + 2x_2 + x_3 \\ \text{s.t} & x_1 + x_2 + 2x_3 \leq 3, \\ & x_1 - x_2 + x_3 \geq 2, \\ & 2x_1 + x_2 + x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

(a) Solve the linear program using the Primal Simplex algorithm.

Solving the program:

Here is the first Simplex Tableau;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	a	b_j	$\frac{b_j}{c_{pj}}$
e_1	1	1	2	1	0	0	0	3	3
a	$1*$	-1	1	0	-1	0	1	2	2
e_3	2	1	1	0	0	1	0	4	2
RC	$3 + M^*$	$2 - M$	$1 + M$	0	$-M$	0	0	$Z = -2M$	

Here is the second Simplex Tableau;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	a	b_j	$\frac{b_j}{c_{pj}}$
e_1	0	2	1	1	1	0	-1	1	$\frac{1}{2}$
x_1	1	-1	1	0	-1	0	1	2	—
e_3	0	3	-1	0	2	1	-2	0	0
RC	0	5	-2	0	3	0	$-(3 + M)$	$Z = 6$	

Even with another iteration the solution will not be improved; $Z^* = 6$, $x_1^* = 2$, $e_1^* = 1$ and all other variables equal to 0.

Here is the last Simplex Tableau ;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	a	b_j	$\frac{b_j}{c_{pj}}$
e_1	0	0	$\frac{5}{3}$	1	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{1}{2}$
x_1	1	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	2	—
x_2	0	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	0
RC	0	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{5}{3}$	< 0	$Z = 6$	

(b) Solve the linear program using the Dual Simplex algorithm.

Here is the first Dual Simplex Tableau;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	b_j
e_1	1	1	2	1	0	0	3
e_2	-1	1	-1^*	0	1	0	-2^*
e_3	2	1	1	0	0	1	4
RC	3	2	1^*	0	0	0	

Here is the second Dual Simplex Tableau;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	b_j
e_1	-1^*	3	0	1	2	0	-1^*
x_3	1	-1	1	0	-1	0	2
e_3	1	2	0	1	1	1	2
RC	2^*	3	0	0	1	0	

Here is the third Dual Simplex Tableau;

Basis	x_1	x_2	x_3	e_1	e_2	e_3	b_j
x_1	1	-3	0	-1	-2	0	1
x_3	0	2	1	1	1	0	1
e_3	0	5	0	$1*$	3	1	1
RC	0	9	0	$2*$	5	0	$Z = 4$

The solution is primal feasible but not optimal as the reduced cost are not all negative.

Here is the first Simplex Tableau;

The solution is optimal.

Basis	x_1	x_2	x_3	e_1	e_2	e_3	b_j
x_1	1	2	0	0	1	1	2
x_3	0	-3	1	0	-2	-1	0
e_1	0	5	0	1	3	1	1
RC	0	-1	0	0	-1	0	$Z = 6$

Exercice 3

Consider the following nonlinear program:

$$\min_{x,y} f(x,y) = x^2 - 3x^2y^2 + 2y^2$$

where x and y are real decision variables.

(a) Give the first order optimality conditions and find **all points** satisfying these conditions.

$\frac{\partial f}{\partial x} = 2x - 6xy^2 = 0$

$\frac{\partial f}{\partial y} = -6x^2y + 4y = 0$

There are 5 points satisfying these conditions: $(0,0)$, $(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}})$, $(-\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}})$, $(\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}})$ and $(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}})$.

(b) Give the second order optimality conditions.

$d^t \nabla^2 f(x,y) d \geq$ to have a minimum.

(c) Conclude on the optimality of each point found in (a).

The only relative minimum here is $(0,0)$ as $d^t \nabla^2 f(0,0) d = 2d_1^2 + 4d_2^2 > 0$.