King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 302 Exam I

Semester (101) November 3, 2010 Time: 05:30 - 07:00 pm

Name: .... I.D: .... Section: ....

Problem	Points
1	/ 15
2	/ 10
3	/ 15
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Total	/ 50

Problem 1. Let α be a real number and A be the matrix defined by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & \alpha \\ 1 & 1 & \alpha^2 \end{pmatrix}.$$

(a) Show that A is row-equivalent to

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 0 & \alpha^2-1 \end{pmatrix}.$$

(b) Discuss the rank of A according to the values of α.

(c) For $\alpha = 1$, find a basis of the solution space of the homogeneous system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution.

(a) We do the following elementary row-operations:

$$A \xrightarrow{R_{12}} \begin{pmatrix} 1 & 0 & \alpha \\ 1 & 1 & 1 \\ 1 & 1 & \alpha^2 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 1 & \alpha^2-\alpha \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 0 & \alpha^2-1 \end{pmatrix},$$

as desired.

(b) – If $\alpha = \pm 1$, then $\text{rank}(A) = 2$.

– If $\alpha \neq \pm 1$, then $\text{rank}(A) = 3$.

(c) The system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 1-\alpha \\ 0 & 0 & \alpha^2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is to say $\begin{cases} x_1 + x_3 = 0 \\ x_2 = 0 \end{cases}$

Hence $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Therefore,

$\mathcal{B} = \{(1, 0, -1)\}$ is a basis of the space of solutions of the homogeneous system $AX = 0$.

Problem 2. Let $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

(a) Find the inverse of M.

(b) Solve the nonhomogeneous system

$$M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution.

(a) We reduce the matrix $B = [M \vdots I_3]$

$$[M \vdots I_3] = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 1 & 1 & 4 & \vdots & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 1 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 1 & 1 & 4 & \vdots & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -1 & 0 \\ 0 & 1 & 2 & \vdots & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 3 & \vdots & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & 2 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & 1 & -1 & 0 \\ 0 & 0 & 1 & \vdots & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

$$\xrightarrow[R_2+R_3]{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & \vdots & \frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & 0 & \vdots & \frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

Consequently, $M =$ is invertible and and

$$M^{-1} = \begin{pmatrix} \frac{2}{3} & 1 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}.$$

(b) The system

$$M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 3. Consider the matrix $A = \begin{pmatrix} 3 & 0 & k \\ 0 & 3 & 3 \\ -1 & -3 & -3 \end{pmatrix}$.

(a) Find all the real values of k for which A is not diagonalizable.
(b) For $k = 1$, find all the eigenvalues of A.

Solution. The characteristic polynomial of A is

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 3 & 0 & -k \\ 0 & \lambda - 3 & -3 \\ 1 & 3 & \lambda + 3 \end{vmatrix} \\ &= (\lambda - 3) \begin{vmatrix} \lambda - 3 & -3 \\ 3 & \lambda + 3 \end{vmatrix} + (-k) \begin{vmatrix} 0 & \lambda - 3 \\ 1 & 3 \end{vmatrix} \\ &= (\lambda - 3)((\lambda - 3)(\lambda + 3) + 9) - k(-(\lambda - 3)) \\ &= (\lambda - 3)(\lambda^2 - 9 + 9 + k) \\ &= (\lambda - 3)(\lambda^2 + k). \end{aligned}$$

Hence the eigenvalues of A are 3, $\pm\sqrt{-k}$, where $\sqrt{-k}$ is a complex number such that its square is $-k$.

We know that, if the eigenvalues of A are pairwise distinct, then A is diagonalisable. So, in order to get A not diagonalisable, it is necessary that one of its eigenvalues is multiple(repeated). In our case, we must have $k = -9$ or $k = 0$.

• Suppose that $k = 0$. Then 0 is a double eigenvalue (of multiplicity 2). Let us find the dimension of the eigenspace $\ker(A - 0I_3)$ associated with 0.

It suffices to find the rank of $A - 0I_3$. Hence, we will reduce the matrix $A - 0I_3 = A$.

By performing the following elementary row-operations

(1) $\frac{1}{3}R_1$, $\frac{1}{3}R_2$
(2) $R_3 + R_1$
(3) $-\frac{1}{3}R_3$
(4) $R_3 - R_2$

This leads to the following reduced matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Consequently, the dimension of the eigenspace $\ker(A-0.I_3)$ is $3-\text{rank}(A-0.I_3) = 3-2 = 1$, which does not equal to the multiplicity of 0. This implies that A is not diagonalisable.

• Suppose that $k = -9$. Then 3 is a double eigenvalue (of multiplicity 2) of A. In order to find the dimension of the eigenspace $\ker(A-3I_3)$ associated with 3, it suffices to reduce $A - 3I_3$.

$$A - 3I_3 = \begin{pmatrix} 0 & 0 & -9 \\ 0 & 0 & 3 \\ -1 & -3 & -6 \end{pmatrix}.$$

Just perform the following elementary row-operations

(1) R_{13}
(2) $-\frac{1}{9}R_3$, $\frac{1}{3}R_2$, $(-1)R_1$
(3) $R_3 - R_2$
(4) $R_1 - 6R_2$

to get the following reduced matrix $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Therefore, $\text{rank}(A - 3I_3) = 2$, and consequently $\dim(\ker(A - 3I_3)) = 3 - 2 = 1$, which is not equal to the multiplicity of 3; showing that A is not diagonalisable.

Of course, if $k \notin \{-9, 0\}$, the the eigenvalues of A are distinct; and consequently, A is diagonalisable.

Conclusion. A is not diagonalisable if and only if $k \in \{-9, 0\}$.

Problem 4.

(a) Let A be an orthogonal and symmetric real matrix. Show that, if λ is an eigenvalue of A then $\lambda = \pm 1$.

(b) Check the above result by considering the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(c) Is A diagonalizable? why?

Solution.

(**a**). Since A is symmetric and orthogonal, we have $A = A^t$ and $AA^t = I_n$. Hence, $A^2 = I_n$. Let λ be an eigenvalue of A; then there exists a non zero vector U of $\mathbb{R}^n$ such that $AU = \lambda U$. Multiplying by A from the left, we get $A^2U = \lambda AU = \lambda(\lambda U) = \lambda^2 U$. But, as $A^2 = I_n$, we get $U = \lambda^2 U$; this leads to $(1 - \lambda^2)U = 0$. Now, as U is a nonzero vector, we deduce that $1 - \lambda^2 = 0$; that is to say $\lambda = \pm 1$.

(**b**). It is easily seen that A is a real symmetric orthogolnal matrix.

The characteristic polynomial of A is

$$\begin{aligned}
P_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda - \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \lambda + \frac{1}{\sqrt{2}} \end{vmatrix} \\
&= (\lambda - 1)((\lambda - \tfrac{1}{\sqrt{2}})(\lambda + \tfrac{1}{\sqrt{2}}) - \tfrac{1}{2}) \\
&= (\lambda - 1)(\lambda^2 - \tfrac{1}{2} - \tfrac{1}{2}) \\
&= (\lambda - 1)(\lambda^2 - 1) \\
&= (\lambda - 1)^2(\lambda + 1).
\end{aligned}$$

Thus the eigenvalues of A are ± 1; and the previous result of (a) is checked.
(**c**). Any real symmetric matrix is diagonalisable.