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Department of Mathematics and Statistics

MATH 302-2

Quiz 3, Term 101

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Exercise. Let $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

Find an orthogonal Matrix Q that diagonalizes A.

SOLUTION

The characteristic polynomial of A is

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = \begin{vmatrix} \lambda-3 & 0 & -1 \\ 0 & \lambda+1 & 0 \\ -1 & 0 & \lambda-3 \end{vmatrix} \\ &= (\lambda+1)\begin{vmatrix} \lambda-3 & -1 \\ -1 & \lambda-3 \end{vmatrix} \\ &= (\lambda+1)((\lambda-3)^2-1) \\ &= (\lambda+1)(\lambda-3-1)(\lambda-3+1) \\ &= (\lambda+1)(\lambda-4)(\lambda-2). \end{aligned}$$

Hence the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 4$.
Now, let us find the eigenspaces associated respectively with λ_1, λ_2 and λ_3
1. $S_1 = \ker(A + I_3)$.
We have to solve the homogeneous system

$$(A+I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The resulting system is

$$\begin{cases} 4x_1 + x_3 = 0 \\ x_1 + 4x_3 = 0 \end{cases}$$

This implies that $x_1 = x_3 = 0$ so that

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus, $U_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a unit eigenvector of A associated with $\lambda_1 = -1$.

2. $S_2 = \ker(A - 2I_3)$.
We solve the homogeneous system

$$(A - 2I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the following system

$$\begin{cases} x_1 & & +x_3 & = & 0 \\ & -3x_2 & & = & 0 \\ x_1 & & x_3 & = & 0 \end{cases}$$

This implies that $x_2 = 0$ and $x_3 = -x_1$, and consequently

$$X = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus, $U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a unit eigenvector of A associated with $\lambda_2 = 2$.

3. $S_3 = \ker(A - 4I_3)$.
We solve the homogeneous system

$$(A - 4I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yield the following system

$$\begin{cases} -x_1 & & +x_3 & = & 0 \\ & -5x_2 & & = & 0 \\ x_1 & & -x_3 & = & 0 \end{cases}$$

that is to say, $x_2 = 0$ and $x_3 = x_1$, and consequently

$$X = \begin{pmatrix} x_1 \\ 0 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a unit eigenvector of A associated with $\lambda_3 = 4$.
We conclude that

$$Q = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

is an orthogonal matrix such that

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$