Section 6.1:

Problem I (B) Let $(x-1)y''+y'=0 \quad (1)$

Find two power series solutions of the given DE (1) about the ordinary point $x=0$

Solution: Let $y=\sum_{n=0}^{\infty} C_n x^n$ So $y'=\sum_{n=1}^{\infty} n C_n x^{n-1}$

and $y''=\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}$.

Substituting in (1) we get:

$$(x-1)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}+\sum_{n=1}^{\infty} nC_n x^{n-1}=0$$

$$\Rightarrow \sum_{n=2}^{\infty} \underbrace{n(n-1)C_n x^{n-1}}_{k=n-1}-\sum_{n=2}^{\infty} \underbrace{n(n-1)C_n x^{n-2}}_{k=n-2}+\sum_{n=1}^{\infty} \underbrace{nC_n x^{n-1}}_{k=n-1}=0$$

$$\Rightarrow \sum_{k=1}^{\infty} k(k+1)C_{k+1}x^k-\sum_{k=0}^{\infty}(k+1)(k+2)C_{k+2}x^k+\sum_{k=0}^{\infty}(k+1)C_{k+1}x^k=0$$

$$\Rightarrow -2C_2+C_1+\sum_{k=1}^{\infty}\left[k(k+1)C_{k+1}-(k+2)(k+1)C_{k+2}+(k+1)C_{k+2}\right]x^k=0$$

Comparing the coefficients, we get:

$$\begin{cases} -2C_2+C_1=0 \\ k(k+1)C_{k+1}-(k+1)(k+2)C_{k+2}=0, \quad k=1,2,3,\ldots \end{cases}$$

$$\Rightarrow C_2=\frac{1}{2}C_1 \quad \text{and} \quad (k+1)C_{k+1}-(k+2)C_{k+2}=0$$