King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 202
Exam II – 2010–2011 (101)
Tuesday, December 07, 2010

Allowed Time: 2 Hours

Name: ______________________________

ID #: ____________________

Section #: __________ Serial Number: __________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have **8 different** problems (8 pages + cover page)

Question #	Grade	Maximum Points
1		11
2		09
3		12
4		08
5		13
6		14
7		12
8		09
Total:		88

11

1. If $y_1 = x\cos(\ln x)$ is a solution of the differential equation:

$$x^2y'' - xy' + 2y = 0,$$

use only the **reduction of order** to find a second solution y_2.

Standard Form: $y'' - \frac{1}{x}y' + \frac{2}{x^2}y = 0$ 02 $\Big|\ P(x) = -\frac{1}{x}$, $f(x) = 0$

$$e^{-\int P(x)\,dx} = e^{\int \frac{dx}{x}} = x \quad 01$$

$$* \ y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx \ \ 02 \ = x\cos(\ln x)\int \frac{x}{x^2\cos^2(\ln x)}\,dx \quad 02$$

Let $t = \ln x$ 01, so

$$\int \frac{dx}{x\cos^2(\ln x)} = \int \frac{dt}{\cos^2 t} = \tan t = \tan(\ln x) \quad 02$$

Hence $y_2(x) = x\cos(\ln x)\cdot\tan(\ln x)$ 01

$$\boxed{y_2(x) = x\sin(\ln x)}$$

09

2. Given that $y_1(x) = -\cos(x)\ln(\sec x + \tan x)$ is a solution of $y'' + y = \tan x$,

03 (a) Find a particular solution of the differential equation

$$y'' + y = 5x + 8.$$

*) 1st method: Clearly $y_p = 5x+8$ is a particular solution of $y''+y = 5x+8$ 03

*) 2nd method: Let $y_p = Ax+B$ be a particular solution of $y''+y=5x+8$ 01

So $y_p' = A$ and $y_p'' = 0 \Rightarrow Ax+B = 5x+B \Rightarrow \begin{cases} A=5 \\ B=8 \end{cases}$ 01

Hence $y_p = 5x+8$ is a particular solution of $y''+y=5x+8$ 01

06 (b) Find a particular solution of the differential equation

$$y'' + y = 2\tan x + 5x + 8.$$

*) $y_{p_1} = -2\cos(x)\operatorname{Ln}(\sec x + \tan x)$ is a particular solution of $y''+y = 2\tan x$. 03

*) From question (a), $y_{p_2} = 5x+8$ is a particular solution of $y''+y=5x+8$

Hence by The Superposition principle, $y_p = y_{p_1} + y_{p_2}$ 03 is a particular solution of:

$$y''+y = 2\tan x + 5x + 8$$

12 / 3. Find the solution to the initial value problem:

$$\begin{cases} y''' + y'' + y' + y = 0 \\ y(0) = 2,\ y'(0) = y''(0) = 0 \end{cases}$$

• Aux. eq: $\begin{cases} m^3 + m^2 + m + 1 = 0 = (m+1)(m^2+1) \\ m_1 = -1 \text{ is a root} \end{cases}$ 02 01

$m_2 = i$, $m_3 = -i$ 01 / $m_2 = \alpha + i\beta$ with $\alpha = 0$ and $\beta = 1$

The General Solution is Then given by:

$$y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$$ 02

*) $y' = -C_1 e^{-x} - C_2 \sin x + C_3 \cos x$ 01

*) $y'' = C_1 e^{-x} - C_2 \cos x - C_3 \sin x$ 01

$$\begin{cases} y(0) = 2 \Rightarrow C_1 + C_2 = 2 \Rightarrow 2C_1 = 2 \\ y'(0) = 0 \Rightarrow -C_1 + C_3 = 0 \Rightarrow C_1 = C_3 \\ y''(0) = 0 \Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2 \end{cases}$$ 02

So $\boxed{C_1 = C_2 = C_3 = 1}$

Hence The General Solution is given by:

$$\boxed{y = e^{-x} + \cos x + \sin x}$$ 02

08

4. Find the **annihilator** of the function

$$8e^{3x}\sin x + x\,e^{-\frac{1}{3}x} + x - 2.$$

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*) For $8e^{3x}\sin x$ $(\alpha = 3, \beta = 1 \text{ and } n = 1)$

So $(D^2 - 6D + 10)(8e^{3x}\sin x) = 0$ 03

*) For $x e^{-\frac{1}{3}x}$ $(n = 2, \alpha = -\frac{1}{3})$, So 02

$$\left(D + \frac{1}{3}\right)^2 \left(x e^{-\frac{1}{3}x}\right) = 0$$

*) For $x - 2$: $D^2(x - 2) = 0$ 01

Hence the annihilator of the function

$8e^{3x}\sin x + x e^{-\frac{1}{3}x} + x - 2$ is:

$$D^2\left(D + \frac{1}{3}\right)^2 (D^2 - 6D + 10)$$ 02

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5. Find the general solution of the differential equation:

$$y'' + 2y' = 4xe^{-2x} + 2\cos x$$

by the **undetermined coefficients** method. **(Do not calculate the constant coefficients of the particular solution)**.

Step I: Find y_c?

H. eq: $y'' + 2y' = 0 \Rightarrow$ Aux. eq: $m^2 + 2m = 0 = m(m+2)$ 01

$\Rightarrow m_1 = 0$ and $m_2 = -2$

So $y_c = C_1 + C_2 e^{-2x}$ 02

Step II Find y_p?

Since $(D+2)^2(4xe^{-2x}) = 0$ 01 and $(D^2+1)(2\cos x) = 0$ 01

Then $(D^2+1)(D+2)^2(D^2+2D)y = 0$ 01

$\Rightarrow D(D+2)^3(D^2+1)y = 0$

Aux. eq: $m(m+2)^3(m^2+1) = 0$ and $\begin{cases} m_1 = 0 \\ m_2 = -2 \text{ (3 tims)}. \\ m_5 = i,\ m_6 = -i \end{cases}$ 02

The General Solution is Then given by:

$$y = \underbrace{C_1 + C_2 e^{-2x}}_{y_c} + C_3 x e^{-2x} + C_4 x^2 e^{-2x} + C_5 \cos x + C_6 \sin x$$ 02

Hence The Basic Form of The particular Solution is

$$y_p = Axe^{-2x} + Bx^2e^{-2x} + C\cos x + D\sin x$$ 02

and The General Solution will be

$$\boxed{y = C_1 + C_2 e^{-2x} + Axe^{-2x} + Bx^2e^{-2x} + C\cos x + D\sin x}$$ 01

14

6. Find the general solution of the differential equation:

$$y'' + 8y' + 16y = \frac{1}{x^2}\, e^{-4x}, \qquad x > 0$$

by using the method of **variation of parameter**.

Step I: y_c? Hom. DE: $y'' + 8y' + 16y = 0$; $f(x) = \frac{1}{x^2} e^{-4x}$ 02

Aux. equation: $m^2 + 8m + 16 = 0 = (m+4)^2$

So $y_c = c_1 e^{-4x} + c_2 x e^{-4x}$ 02

Step II: y_p? Let $y_1 = e^{-4x}$ and $y_2 = x e^{-4x}$

Find y_p such that: $y_p = u_1 y_1 + u_2 y_2$ 01

$$\cdot\ W(y_1, y_2) = \begin{vmatrix} e^{-4x} & x e^{-4x} \\ -4e^{-4x} & e^{-4x} - 4x e^{-4x} \end{vmatrix} = e^{-8x} - 4x e^{-8x} + 4x e^{-8x} = e^{-8x} \neq 0$$ 02

$$\cdot\ u_2' = -\frac{y_2 f(x)}{W} = \frac{-x e^{-4x} \cdot \frac{1}{x^2} e^{-4x}}{e^{-8x}} = -\frac{1}{x} \Rightarrow \boxed{u_1 = -\ln x}$$ 02

$$\cdot\ u_2' = \frac{y_1 f(x)}{W} = \frac{e^{-4x} \cdot \frac{1}{x^2} e^{-4x}}{e^{-8x}} = \frac{1}{x^2} \Rightarrow \boxed{u_2 = -\frac{1}{x}}$$ 02

So $y_p = u_1 y_1 + u_2 y_2 = -e^{-4x} \ln x - e^{-4x} = -e^{-4x}(\ln x + 1)$ 01

Hence The General Solution is:

$$\boxed{y = y_c + y_p = c_1 e^{-4x} + c_2 x e^{-4x} - e^{-4x}(1 + \ln x)}$$ 02

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7. (i) Find the general solution of the differential equation

$$4t^2y'' + 8ty' + y = 0 \tag{1}$$

directly as a **Cauchy-Euler equation**.

Let $y = t^m \Rightarrow y' = mt^{m-1}$, $y'' = m(m-1)t^{m-2}$

substitute in (1): 02

$$4t^2 m(m-1)t^{m-2} + 8tmt^{m-1} + t^m = 0 = t^m[4m(m-1)+8m+1]$$

Aux. eq: $4m(m-1)+8m+1=0$ or $4m^2+4m+1=0$ 01

$$4m^2+4m+1=0 = m^2+m+\frac{1}{4} = \left(m+\frac{1}{2}\right)^2$$ 01

$$m_1 = m_2 = -\frac{1}{2}$$

Hence The General Solution is given by: 02

$$\boxed{y = c_1 t^{-\frac{1}{2}} + c_2 t^{-\frac{1}{2}} \ln t}$$

(ii) Transform the differential equation (1) into a differential equation with constant coefficients **(Do not solve the obtained differential equation)**.

- Let $t = e^x$ (or $x = \ln t$), 01 we have: $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}$ 01
- $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{1}{t}\cdot\frac{dy}{dx}\right) = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$. 02

Substitute in equation (1):

$$4t^2\left[\frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\right] + 8t\left(\frac{1}{t}\cdot\frac{dy}{dx}\right) + y = 0 = 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y$$

which is a DE with Constant Coefficients. 02

09

8. Find the **Cauchy-Euler equation** whose solution is given by:

$$y = c_1 x + c_2 x \cos(\ln x) + c_3 x \sin(\ln x).$$

From the solution, the roots of the auxi- equation are:

$m_1 = 1$ (01), $m_2 = 1+i$ (01) and $m_3 = 1-i$ (01) | (03), Then:

Auxi- eq: $(m-1)(m^2-2m+2) = 0 = (m-1)m(m-2) + 2m - 2$ (02)

Then $m^3 - 3m^2 + 4m - 2 = 0 = m(m-1)(m-2) + 2m - 2$ (01)

Hence the Cauchy-Euler equation is:

$$x^3 y''' + 2xy' - 2y = 0$$ (03)