Math 202 - Elements of Differential Equations
Autumn 2010, Quiz 2
Dr. Salim Belhaiza

Section 4.2
Knowing that $y_1 = e^{2x}$ is a solution to the differential equation
$y'' - 4y = 0$
Use reduction of order to find the second solution.

Solution 4.2
$y_2 = uy_1$ gives $y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + 2u'y_1' + uy_1''$.

Substitution gives : $u''y_1 + 2u'y_1' + uy_1'' - 4uy_1 = u''y_1 + 2u'y_1' + u\underbrace{(y_1'' - 4y_1)}_{0} = 0$.

Then, $y_1' = 2e^{2x}$ gives $u''e^{2x} + 2u'2e^2x = 0 \Rightarrow u'' + 4u' = 0$.

Let $w = u'$,then $w' + 4w = 0 \Rightarrow w = e^{-\int 4dx} = e^{-4x}$.

Thus, $u = \frac{-1}{4}e^{-4x}$ and $y_2 = \frac{-1}{4}e^{-2x}$.

Section 4.3

(a) Solve the following differential equation and give its real valued solutions :

$$y'' + y' + y = 0$$

(b) Use the result found in (a) to find the general solution to the differential equation :

$$y'' + y' + y = e^x$$

Solution 4.3

(a) Assume $y = e^{mx}$, then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$. The auxilary equation obtained is : $m^2 + m + 1 = 0 \Rightarrow \Delta = -3$ and $\sqrt{\Delta} = -\sqrt{3}i$.

Thus $m_1 = \frac{1}{2}(-1 - \sqrt{3}i)$ and $m_2 = \frac{1}{2}(-1 + \sqrt{3}i)$. The real valued solutions to the differential equation are :

$y_1 = e^{-\frac{1}{2}x} cos(\frac{\sqrt{3}}{2}x)$ and $y_2 = e^{-\frac{1}{2}x} sin(\frac{\sqrt{3}}{2}x)$.

(b) Use of the annihilator $(D - 1)$ will make the differential equation in (b) become homogeneous with third order.

The general solution to the differential equation is then given by ;

$$y(x) = c_1 e^x + c_2 y_1 + c_3 y_2$$

After substitution it appears that $c_1 = \frac{1}{3}$.

Section 4.5

(a) We know that $(D-\alpha)$ annihilates $e^{\alpha x}$.

Suppose that $(D-\alpha)^{n-1}$ annihilates $x^{n-2}e^{\alpha x}$ and give the proof that $(D-\alpha)^n$ annihilates $x^{n-1}e^{\alpha x}$.

(b) We know that $(D^2-2\alpha D+(\alpha^2+\beta^2))$ annihilates $e^{\alpha x}cos(\beta x)$.

Suppose that $(D^2-2\alpha D+(\alpha^2+\beta^2))^{n-1}$ annihilates $x^{n-2}e^{\alpha x}cos(\beta x)$ and give the proof that $(D^2-2\alpha D+(\alpha^2+\beta^2))^n$ annihilates $x^{n-1}e^{\alpha x}cos(\beta x)$.

Solution 4.5

(a) $(D-\alpha)^n x^{n-1}e^{\alpha x} = (D-\alpha)^{n-1}(D-\alpha)x^{n-1}e^{\alpha x} = (D-\alpha)^{n-1}\left[(n-1)x^{n-2}e^{\alpha x} + \alpha x^{n-1}e^{\alpha x} - \alpha x^{n-1}e^{\alpha x}\right]$ $(n-1)(D-\alpha)^{n-1}x^{n-2}e^{\alpha x} = (n-1)\times 0 = 0$.

(b) $(D^2-2\alpha D+(\alpha^2+\beta^2))^n x^{n-1}e^{\alpha x}cos(\beta x) = (D^2-2\alpha D+(\alpha^2+\beta^2))^{n-1}(D^2-2\alpha D+(\alpha^2+\beta^2))x^{n-1}e^{\alpha x}cos(\beta x)$.

After calculation :

$(D^2-2\alpha D+(\alpha^2+\beta^2))x^{n-1}e^{\alpha x}cos(\beta x) = D(n-1)x^{n-2}e^{\alpha x}cos(\beta x)-(n-1)\beta x^{n-1}e^{\alpha x}sin(\beta x)$.

Applying $(D^2-2\alpha D+(\alpha^2+\beta^2))^{n-1}$ would yield 0 because $(D^2-2\alpha D+(\alpha^2+\beta^2))^{n-1}$ annihilates $x^{n-2}e^{\alpha x}cos(\beta x)$ and $(D^2-2\alpha D+(\alpha^2+\beta^2))$ annihilates $e^{\alpha x}sin(\beta x)$.

Section 4.6

Solve the following differential equation using the variation of parameters approach.

$y'' - 2y' + y = xe^x$

Solution 4.6

(1) Solve the corresponding homogeneous DE : $y'' - 2y' + y = 0$
with $y = e^{mx} \Rightarrow m^2 - 2m + 1 = 0$,
$\Delta = 0 \Longrightarrow m_1 = m_2 = 1 \Longrightarrow y_1 = e^x$ and $y_2 = xe^x$.

(2) Using variation of parameters, $y_p = u_1 y_1 + u_2 y_2$.

$u_1' = \frac{w_1}{w}$, $u_2' = \frac{w_2}{w}$ such that :

With $W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$,

$W_1 = \begin{vmatrix} 0 & xe^x \\ xe^x & e^x + xe^x \end{vmatrix} = -x^2 e^{2x} \Rightarrow u_1' = -x^2 \Rightarrow u_1 = -\frac{x^3}{3}$.

And $W_2 = \begin{vmatrix} e^x & 0 \\ e^x & xe^x \end{vmatrix} = xe^{2x} \Rightarrow u_2' = -x \Rightarrow u_2 = \frac{x^2}{2}$.

(3) $y_p =_1 y_1 + u_2 y_2 = -\frac{x^3}{3} e^x + \frac{x^3}{2} e^x = \frac{x^3}{6} e^x$.

$\Rightarrow y = c_1 e^x + c_2 x e^x + \frac{x^3}{6} e^x$.

Section 4.7
Solve the following differential equation.
$4x^2y'' - 6xy' + 6y = 4x^3$

Solution 4.7
First, we find the solutions to the corresponding homogeneous differential equation :
$4m^2 - 10m + 6 = 0$
$\Delta = 25 - 24 = 1 \Rightarrow \sqrt{\Delta} = 1$
$m_1 = 1$ and $m_2 = \frac{3}{2} \Rightarrow y_c = C_1 x + C_2 x^{\frac{3}{2}}$.
Dividing the differential equation by $4x^2$ yields $y'' - \frac{3}{2x}y' + \frac{3}{2x^2}y = x$.
Then, using the assumption : $y_p = u_1 y_1 + u_2 y_2$,
we find $u_1' = \frac{w_1}{w}$ and $u_2' = \frac{w_2}{w}$.
With $W = \begin{vmatrix} x & x^{\frac{3}{2}} \\ 1 & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = \frac{1}{2}x^{\frac{3}{2}}$,
$W_1 = \begin{vmatrix} 0 & x^{\frac{3}{2}} \\ x & \frac{3}{2}x^{\frac{1}{2}} \end{vmatrix} = -x^{\frac{5}{2}}$,
and $W_2 = \begin{vmatrix} x & 0 \\ 1 & x \end{vmatrix} = x^2$.
Hence, $u_1' = \frac{-x^{\frac{5}{2}}}{\frac{1}{2}x^{\frac{3}{2}}} = -2x$ and $u_2' = \frac{x^2}{\frac{1}{2}x^{\frac{3}{2}}} = 2x^{\frac{1}{2}}$.
Thus $u_1 = \int -2x dx = -x^2$.
and $u_2 = \int 2x^{\frac{1}{2}} dx = \frac{4}{3}x^{\frac{3}{2}}$.
The particular solution is then $y_p = \frac{1}{3}x^3$.
Finally $y = \frac{1}{3}x^3 + C_1 x + C_2 x^{\frac{3}{2}}$.