King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 202
Final Exam – 2010–2011 (101)
Monday, January 24, 2011

Allowed Time: 3 Hours

Name: ____________________________

ID #: ____________________

Section #: __________ Serial Number: __________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have **5 different MCQ** problems.
5. Make sure that you have **5 written** problems.

Part I: MCQ Problems

Question	Answer	Grade	Maximum Grade
1			12
2			12
3			12
4			12
5			12
Total:			60

Part II: Written Problems

Question #	Grade	Maximum Points
6		20
7		22
8		15
9		15
10		12
Total:		84

MCQ Problems:

Answer Counts:

Q		V1	V2	V3	V4
1		b	e	a	a
2		c	b	b	c
3		a	d	c	d
4		c	e	b	d
5		c	c	c	c

20

6. Find the recurrence relation determining the coefficients of the power series solution:

$$y'' - xy' + 3y = 0$$

about the point $x = 0$.

- Clearly 0 is an ordinary point. 02
- Let $y = \sum_{n=0}^{\infty} C_n x^n$, so $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ 02

02

Thus $y'' - xy' + 3y = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=0}^{\infty} 3 C_n x^n = 0$

04

So $\sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} x^k - \sum_{k=1}^{\infty} k C_k x^k + 3C_0 + 3\sum_{k=1}^{\infty} C_k x^k = 0$

03

$2C_2 + 3C_0 + \sum_{k=1}^{\infty} \left[(k+1)(k+2) C_{k+2} - k C_k + 3C_k \right] x^k = 0$

04

$$\Rightarrow \begin{cases} 3C_0 + 2C_2 = 0 \\ (k+1)(k+2) C_{k+2} + (3-k) C_k = 0, \quad k = 1, 2, \dots \end{cases}$$

03

Hence $\begin{cases} C_2 = -\frac{3}{2} C_0 \\ C_{k+2} = \frac{(k-3)}{(k+1)(k+2)} C_k, \quad k = 1, 2, 3, \dots \end{cases}$

22

7. Consider the following differential equation:

$$3xy'' + 2y' - y = 0.$$

(a) Is there any singular point associated to the differential equation? If yes, find it.

(b) Give the indicial equation and its roots.

(c) Find the power series solution associated to the largest indicial root.

a. The standard form is $y'' + \frac{2}{3x}y' + \frac{1}{3x}y = 0$. The only singular point is $x_0 = 0$. With $P(x) = \frac{2}{3x}$, we have $xP(x) = \frac{2}{3}$ which is analytic at 0. Also with $Q(x) = \frac{1}{3x}$, we have $x^2Q(x) = \frac{x}{3}$ which is also analytic at 0.
Thus $x_0 = 0$ is a regular singular point of the differential equation. 03

b. With the assumption that $y = \sum_{n=0}^{\infty} C_n x^{n+r}$, the indicial equation is $r(r-1)+\frac{2}{3}r = 0$. The two roots of this equation are $r_1 = \frac{1}{3}$ and $r_2 = 0$. Note that $r_1 - r_2 = \frac{1}{3}$. 03

c. Using the assumption $y = \sum_{n=0}^{\infty} C_n x^{n+r}$ and subsituting we will have :

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)C_n x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r} = 0.$$

02

To unify the power of x, we let $p = n+1$ so $\sum_{n=0}^{\infty} C_n x^{n+r} = \sum_{p=1}^{\infty} C_{p-1} x^{p+r-1}$.

03

$$\Rightarrow \sum_{n=0}^{\infty} 3(n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)C_n x^{n+r-1} + \sum_{p=1}^{\infty} C_{p-1} x^{p+r-1} = 0.$$

$$\Rightarrow [3r(r-1)+2r]\,C_0 x^{r-1} + \sum_{n=1}^{\infty} x^{n+r-1}\,[3(n+r)(n+r-1)C_n + 2(n+r)C_n + C_{n-1}] = 0.$$

$$\Rightarrow [3r(r-1)+2r] = 0 \quad \text{and} \quad (n+r)(3n+3r-1)C_n + C_{n-1} = 0, \quad \text{for} \quad n = 1, 2, .., \infty.$$

03

Which means that :
$C_n = -\frac{C_{n-1}}{(n+r)(3n+3r-1)}$ for $n = 1, 2, .., \infty$. 02

Since $r_1 = \frac{1}{3}$ is The Largest indicial root we will have:

(02)

$$\boxed{c_n = -\frac{c_{n-1}}{(3n^2+n)} \text{ for } n = 1, 2, .., \infty.}$$

$n = 1$ gives $c_1 = -\frac{c_0}{4}$.

$n = 2$ gives $c_2 = -\frac{c_1}{14} = \frac{c_0}{4\times 14}$.

$n = 3$ gives $c_3 = -\frac{c_0}{4\times 14\times 30}$.

$n = 4$ gives $c_4 = \frac{c_0}{4\times 14\times 30\times 52}$.

Thus $y_1(x) = c_0 x^{\frac{1}{3}}\left[1 - \frac{x}{4} + \frac{x^2}{4\times 14} - \frac{x^3}{4\times 14\times 30} + \frac{x^4}{4\times 14\times 30\times 52} + ...\right]$.

(04)

15

8. Find the solution of the differential equation:

$$X' = AX\,; \quad A = \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix}.$$

- characteristic equation: $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -3 \\ 3 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 9 = \lambda^2 - 6\lambda + 18$$ 02

02

So The eigenvalues of A are $\lambda_1 = 3+3i$ and $\lambda_2 = \overline{\lambda_1} = 3-3i$

- Let $K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ be The eigenvector associated to λ_1, so 03

$$(A - \lambda_1 I)K = 0 \Rightarrow \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3ik_1 - 3k_2 = 0 \\ 3k_1 - 3i\,k_2 = 0 \end{cases}$$

$\Rightarrow k_1 = i k_2$, for $k_2 = 1 \Rightarrow k_1 = i$ 02.

and $K = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$X_1 = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 3t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 3t\right)e^{3t} = \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}e^{3t}$$

03

$$X_2 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 3t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 3t\right)e^{3t} = \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}e^{3t}$$

Hence $X = c_1 X_1 + c_2 X_2 = \begin{pmatrix} -c_1 \sin 3t + c_2 \cos 3t \\ c_2 \cos 3t + c_2 \sin 3t \end{pmatrix} e^{3t}$ 03

15

9. Consider the nonhomogeneous system:

$$X' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} X + F(t)\,; \quad \text{where} \quad F(t) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Let $\Phi(t) = \begin{pmatrix} 2 & 6e^t \\ 2 & 4e^t \end{pmatrix}$ be the fundamental matrix of the associated homogeneous system. Use **variation of parameters** method to find a particular solution X_p, and form the general solution.

$$\det \Phi = -4e^t \neq 0$$

$$\Phi^{-1} = -\frac{1}{4e^t}\begin{pmatrix} 4e^t & -2 \\ -6e^t & 2 \end{pmatrix}^T = -\frac{1}{2}\begin{pmatrix} 2 & -3 \\ -e^{-t} & e^{-t} \end{pmatrix}$$

03

- $$U(t) = \int \Phi^{-1} F(t)\,dt = -\frac{1}{2}\int \begin{pmatrix} 12 \\ -5e^{-t} \end{pmatrix} dt = -\frac{1}{2}\begin{pmatrix} 12t \\ 5e^{-t} \end{pmatrix}$$

04

So $$X_p = \Phi(t)\,U(t) = -\frac{1}{2}\begin{pmatrix} 2 & 6e^t \\ 2 & 4e^t \end{pmatrix}\begin{pmatrix} 12t \\ 5e^{-t} \end{pmatrix} = \begin{pmatrix} -15-12t \\ -10-12t \end{pmatrix}$$

04

The General Solution is:

$$X(t) = X_c + X_p = c_1\begin{pmatrix} 2 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 6e^t \\ 4e^t \end{pmatrix} + \begin{pmatrix} -15-12t \\ -10-12t \end{pmatrix}$$

$$= \begin{pmatrix} 2c_1 + 6c_2e^t - 12t - 15 \\ 2c_1 + 4c_2e^t - 12t - 10 \end{pmatrix}.$$

04

12

10. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$.

(a) Show that $A^3 = 0$.

(b) Compute e^{At}.

(c) Use e^{At} to find the general solution of the homogeneous linear system:

$$X' = AX.$$

a/ $A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ 01

$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 01

b) $e^{At} = I + At + A^2 \frac{t^2}{2}$ 02

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3t & 0 & 0 \\ 5t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{2}t^2 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2+5t & t & 1 \end{pmatrix}$

03

c/ $X = e^{At}C = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2+5t & t & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ 03

$= \begin{pmatrix} c_1 \\ 3c_1t + c_2 \\ \frac{3}{2}c_1t^2 + 5c_1t + c_2t + c_3 \end{pmatrix}$ 02