MATH 202.1 (101)

Quiz 2 (Chap. 2.4-3.1) Duration: 20mn

Name: ID number:

1.)(5pts)Find an integrating factor that makes the following DE exact and solve the resulting equation: $(x^2+y^2-5)dx-(y+xy)dy=0.$

2.)(5pts)Solve the homogeneous DE: $xydy-(3y^2+x^2)dx=0.$

Solution

1) $M_y = 2y$, $N_x = -y$

$\frac{M_y - N_x}{N} = -\frac{3y}{y(1+x)} = \frac{-3}{1+x}$

An integrating factor is $e^{-\int \frac{3}{1+x}dx}$

which is $e^{-3\ln|1+x|} = \frac{1}{(1+x)^3}$, $x>-1$

Multiplying the equation by $\frac{1}{(1+x)^3}$, we get

$\frac{(x^2+y^2-5)}{(x+1)^3}dx - \frac{y}{(1+x)^2}dy = 0$, $x>-1$

For this new equation, we have

$M_y = \frac{2y}{(x+1)^3}$, $N_x = \frac{2y}{(1+x)^3}$.

This new equation is exact.

There exists $f(x,y)$ such that

$\frac{\partial f}{\partial x} = \frac{x^2+y^2-5}{(x+1)^3}$, $\frac{\partial f}{\partial y} = \frac{-y}{(1+x)^2}$

This implies $f(x,y) = \frac{-y^2}{2(1+x)^2} + g(x)$,

and, $\frac{y^2}{(1+x)^3} + g'(x) = \frac{x^2+y^2-5}{(1+x)^3}$

$\Rightarrow g'(x) = \frac{x^2-5}{(1+x)^3} = \frac{a}{1+x} + \frac{b}{(1+x)^2} + \frac{c}{(1+x)^3}$

$= \frac{1}{1+x} - \frac{2}{(1+x)^2} - \frac{4}{(1+x)^3}$

$\Rightarrow g(x) = \int \frac{1}{1+x}dx - 2\int \frac{1}{(x+1)^2}dx - 4\int \frac{1}{(1+x)^3}dx$

$= \ln(1+x) + \frac{2}{x+1} + \frac{2}{(1+x)^2}$, $x>-1$

$$\boxed{\frac{-y^2}{2(1+x)^2} + \ln(1+x) + \frac{2}{1+x} + \frac{2}{(1+x)^2} = C, \quad x>-1}$$

2) $M(tx,ty) = t^2 M(x,y)$

$N(tx,ty) = t^2 N(x,y)$

We consider the substitution $y = ux$

$\Rightarrow dy = xdu + u dx$.

Substituting y into the equation, we get

$x^2u[xdu + udx] - (3x^2u^2+x^2)dx = 0$

$\frac{u}{2u^2+1}du - \frac{1}{x}dx = 0$, $x>0$

We integrate, and find

$\int \frac{u}{2u^2+1}du - \int \frac{dx}{x} = 0$,

$\frac{1}{4}\ln(2u^2+1) - \ln x = C$

$\ln(2u^2+1) - 4\ln x = C$

$\frac{2u^2+1}{x^4} = C$

$\frac{2(\frac{y}{x})^2+1}{x^4} = C$

$$\boxed{\frac{2y^2+x^2}{x^6} = C, \quad x>0}$$