King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 202
Exam I – 2010–2011 (101)
Tuesday, November 02, 2010

Allowed Time: 2 Hours

Name: ______________________

ID #: ______________

Section #: ________ Serial Number: ________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have **9 different** problems (9 pages + cover page)

Question #	Grade	Maximum Points
1	10	
2	09	
3	09	
4	10	
5	12	
6	11	
7	09	
8	10	
9	08	
Total:		88

04 1. (a) Verify that $e^y = y - x^2 + C$ is an implicit solution of the differential equation:

$$\frac{dy}{dx} = \frac{2x}{1-e^y}.$$

$e^y = y - x^2 + C$: differentiate with respect to y:

$$e^y \frac{dy}{dx} = \frac{dy}{dx} - 2x \quad 01, \quad \text{So} \quad \frac{dy}{dx}\left(1 - e^y\right) = 2x \quad 01$$

Thus $\dfrac{dy}{dx} = \dfrac{2x}{1-e^y}$ 01

Hence $e^y = y - x^2 + C$ is an implicit Solution of The given (DE). 01

06 (b) Determine the region in which the differential equation:

$$y' = \frac{\sqrt{y^2-1}}{x}$$

has a unique solution through the point $(1, -2)$.

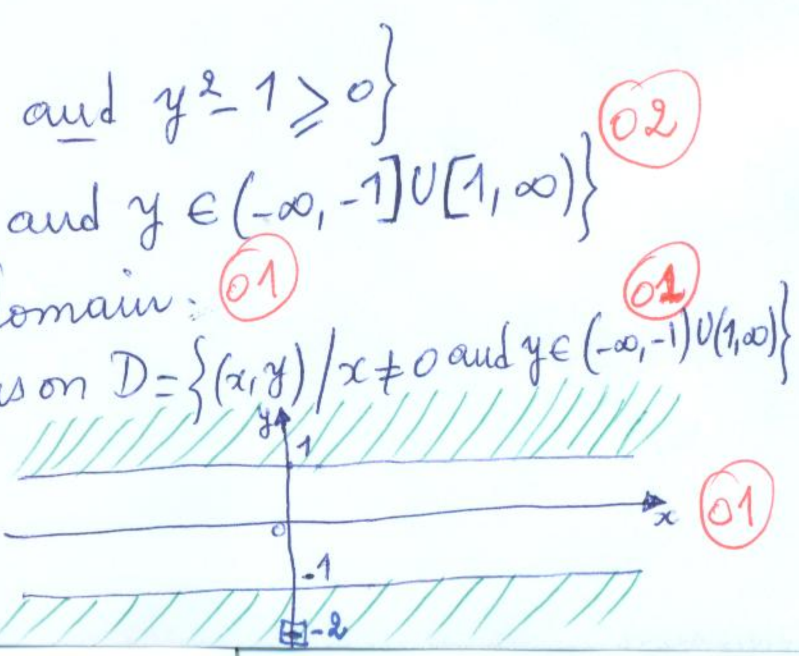
- $f(x,y) = \dfrac{\sqrt{y^2-1}}{x}$
- $D_f = \left\{(x,y) / x \neq 0 \text{ and } y^2 - 1 \geq 0\right\}$ 02
  $= \left\{(x,y) / x \neq 0 \text{ and } y \in (-\infty, -1] \cup [1, \infty)\right\}$

i/. f is Continuous on its domain. 01

ii/. $\dfrac{\partial f}{\partial y} = \dfrac{y}{x\sqrt{y^2-1}}$ is Continuous on $D = \left\{(x,y) / x \neq 0 \text{ and } y \in (-\infty,-1) \cup (1,\infty)\right\}$ 01

iii/. $(1,-2) \in D$ 01

$\Longrightarrow$ The region is Then D:



03 2. (a) Find the critical points of the differential equation:

$$\frac{dy}{dx} = y^3 - y. \tag{1}$$

$f(y) = y^3 - y$. $\quad y^3 - y = 0 \Rightarrow y(y^2-1) = 0$ 01

$\Rightarrow y(y-1)(y+1) = 0$ 01

$\Rightarrow y = 0,\ y = 1,\ y = -1$

Hence the critical points are: 0, 1 and -1 01

06 (b) Draw the portrait line of the differential equation (1) and then classify the critical points determined in (a).

		-1		0		1	
$y+1$	$-$	0	$+$		$+$		$+$
y	$-$		$-$	0	$+$		$+$
$y-1$	$-$		$-$		$-$	0	$+$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

03

Unstable 01

03

stable 01

Unstable 01

09

3. Solve the initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{x-x^2} \\ y\left(\dfrac{1}{2}\right) = 1\,. \end{cases}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x-x^2} \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x-x^2}$$ 02

$$\Rightarrow \sin^{-1} y = \int \frac{dx}{x(1-x)}$$ 01

So $$\sin^{-1} y = \int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx$$

$$\Rightarrow \sin^{-1} y = \ln|x| - \ln|1-x| + c$$ 02

$$\Rightarrow \sin^{-1} y = \ln\left|\frac{x}{1-x}\right| + c$$ 01

given $y\left(\frac{1}{2}\right) = 1 \Rightarrow \sin^{-1} 1 = \ln\left(\frac{1/2}{1-1/2}\right) + c = \ln 1 + c$

$$\Rightarrow \boxed{c = \frac{\pi}{2}}$$ 02

Hence $$\boxed{\sin^{-1} y = \ln\left|\frac{x}{1-x}\right| + \frac{\pi}{2}}$$ 01

10

4. Solve the differential equation:

$$xy' - y + e^{1/x} = 0. \qquad (1)$$

Standard form: (2) $y' - \frac{1}{x}y = -\frac{1}{x}e^{1/x}$ 02 ; $P(x) = -\frac{1}{x}$ 01

Integrating Factor:

$$e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x} \quad 02$$

Multiplying the (DE)(2) by the I.F gives:

$$\frac{1}{x}y' - \frac{1}{x^2}y = -\frac{1}{x^2}e^{\frac{1}{x}} = \frac{d}{dx}\left(\frac{1}{x}y\right) \quad 02$$

So $\frac{1}{x}y = -\int \frac{1}{x^2}e^{\frac{1}{x}}\,dx = e^{\frac{1}{x}} + C$ 02

Hence $\boxed{y = x\left(e^{\frac{1}{x}} + C\right)}$ 01

12 5. (a) Solve the differential equation:

$$(3x^5 + 2x^3y^2 - 20x^3)dx + x^4y\,dy = 0.$$

$M(x,y) = 3x^5 + 2x^3y^2 - 20x^3 \Rightarrow \frac{\partial M}{\partial y} = 4x^3y$ The equation

$N(x,y) = x^4y \Rightarrow \frac{\partial N}{\partial x} = 4x^3y$ is Exact. 02

So There exist f / $\frac{\partial f}{\partial y} = x^4y$ and $\frac{\partial f}{\partial x} = 3x^5 + 2x^3y^2 - 20x^3$

- $\frac{\partial f}{\partial y} = x^4y \Rightarrow f(x,y) = \frac{1}{2}x^4y^2 + g(x)$, 02
- $\frac{\partial f}{\partial x} = 2x^3y^2 + g'(x) = 2x^3y^2 + 3x^5 - 20x^3 \Rightarrow g'(x) = 3x^5 - 20x^3$ 02

Thus $g(x) = \frac{1}{2}x^6 - 5x^4 + c$, Hence a Solution of The given (DE) is:

$$\boxed{\frac{x^4y^2}{2} + \frac{1}{2}x^6 - 5x^4 = C}$$ 01

(b) Consider the differential equation:

$$(xy\cos x - 2y\sin x)dx + 2x\sin x\,dy = 0. \quad (1)$$

(i) Determine whether the differential equation (1) is Exact or not.

- $M(x,y) = xy\cos x - 2y\sin x \Rightarrow \frac{\partial M}{\partial y} = x\cos x - 2\sin x$
- $N(x,y) = 2x\sin x \Rightarrow \frac{\partial N}{\partial x} = 2\sin x + 2x\cos x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, The equation is not Exact. 02

(ii) In case that the differential equation (1) is **not Exact**, find an integrating factor which makes equation (1) **exact**. **(Do not solve the obtained equation)**

$$\frac{M_y - N_x}{N} = \frac{x\cos x - 2\sin x - 2\sin x - 2x\cos x}{2x\sin x} = -\frac{1}{2}\frac{\cos x}{\sin x} - \frac{2}{x}$$

which depend only on x, Thus an integrating factor is given by:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx} = e^{\ln\left(\frac{1}{x^2\sqrt{|\sin x|}}\right)} = \frac{1}{x^2\sqrt{|\sin x|}}$$ 03

$$\boxed{\mu(x) = \frac{1}{x^2\sqrt{|\sin x|}}}$$

11

6. (a) Find a suitable substitution that transforms the differential equation

$$\sqrt{y}\,dy = (x - y^{3/2})dx,$$

into a **linear** differential equation. Find the new linear equation but **do not solve it**.

$\sqrt{y}\,\frac{dy}{dx} = x - y^{3/2} \Rightarrow \frac{dy}{dx} = x y^{-1/2} - y^{-1}$

$\Rightarrow \frac{dy}{dx} + y = x y^{-1/2}$, This is a 02

Bernoulli equation (with $n = -\frac{1}{2}$)

Let $u = y^{1-n} = y^{1+\frac{1}{2}} = y^{3/2}$ 02, So $\frac{du}{dx} = \frac{3}{2} y^{1/2} \frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} = \frac{3}{2}(x - u) \Rightarrow \boxed{\frac{du}{dx} + \frac{3}{2}u = \frac{3}{2}x}$ which

01

is a Linear DE.

(b) Find a suitable substitution that transforms the differential equation

$$y' = \sqrt{7x + y} - 7$$

into a **separable** differential equation. Find the new separable equation but **do not solve it**.

01

Let $u = 7x + y$. Then $\frac{du}{dx} = 7 + \frac{dy}{dx}$ 02

So $\frac{dy}{dx} = \frac{du}{dx} - 7 = \sqrt{u} - 7$

$\Rightarrow \frac{du}{dx} = u^{\frac{1}{2}}$ 02

$\Rightarrow \frac{du}{u^{1/2}} = dx$ which is a Separable DE. 01

09

7. A glass of water initially at 70° F is placed in a freezer. The freezer is kept at the constant temperature 50° F. After one hour the temperature of the water in the glass is 60° F. Find the exact time needed for the temperature of the water to reach 52° F after it is placed in the freezer.

We have $\frac{dT}{dt} = k(T-50)$, $T(0) = 70$ 02

$\Rightarrow T(t) = 50 + Ce^{kt}$

Using $T(0) = 70 \Rightarrow \boxed{C = 20}$ 01

So $T(t) = 50 + 20e^{kt}$

• $T(1) = 60 \Rightarrow 60 = 50 + 20e^{k}$

$\Rightarrow 10 = 20e^{k} \Rightarrow e^{k} = \frac{1}{2}$

$\Rightarrow \boxed{k = \ln\frac{1}{2} = -\ln 2}$ 02

• $T(t) = 50 + 20e^{-t\ln 2} = 50 + 20\left(\frac{1}{2}\right)^{t}$ 02

Now For $T(t) = 52$ We have:

$$52 = 50 + 20\left(\frac{1}{2}\right)^{t}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{t} = \frac{1}{10}$$

Thus $\boxed{t = \frac{\ln 10}{\ln 2}}$ 02

10

8. (a) Verify that $y = c_1 e^t \sin 2t + c_2 e^t \cos 2t$ is a two-parameter family of solutions of the differential equation:

$$y'' - 2y' + 5y = 0. \tag{1}$$

$y' = c_1 e^t(\sin 2t + 2\cos 2t) + c_2 e^t(\cos 2t - 2\sin 2t)$ 02

$y'' = c_1 e^t(-3\sin 2t + 4\cos 2t) + c_2 e^t(-3\cos 2t - 4\sin 2t)$ 02

So $y'' - 2y' + 5y = 0$ 01, Thus

$y = c_1 e^t \sin 2t + c_2 e^t \cos 2t$ is a two-parameter family of Solutions of The DE (1). 01

(b) Determine whether a member of the family of solutions of the differential equation (1) can be found that satisfies the boundary conditions:

$$y(0) = 1, \quad y(\pi) = -1.$$

- $y(0) = 1 = c_1 \times 0 + c_2 \Rightarrow c_2 = 1$ 01
- $y(\pi) = -1 = c_1 e^{\pi} \times 0 + c_2 e^{\pi} = e^{\pi}$: impossible. 01

Thus The Boundary value problem has no Solution. 02

08

9. Consider the differential equation

$$y'' - 4y' + 4y = 0. \qquad (1)$$

(a) Find the interval in which the two solutions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are linearly independent.

02

• y_1 and y_2 are Linearly independent $\Longleftrightarrow W(y_1, y_2) \neq 0$

03

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x} \neq 0$$

for every x.

Hence The interval is ; $I = (-\infty, \infty)$. 01

(b) Form a general solution for the differential equation (1).

$$y = c_1 y_1 + c_2 y_2 .$$

02